Efficiently deciding if an ideal is toric after a linear coordinate change

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June 2, 2025

Definition. A prime ideal $I \subset k[x_1, ..., x_n]$ is **toric** if it can be generated by pure differences of monomials:

$$I = \langle x^u - x^v \mid u, v \in \mathbb{N}^n \rangle$$

Question: Given a prime ideal $I \subset k[x_1, ..., x_n]$, does there exist a linear coordinate change $g \in \mathrm{GL}_n(k)$ such that $g.I$ is toric?

- Is this decidable on a Turing machine?
- Can such a g be computed? Efficiently?

Why?

- **Sparse equations are better than dense equations.**
  - Is I generated by binomials?
  - Does I contain a binomial?
  - Is I generated by binomials after linear coordinate change?
  - Does I contain a binomial after a linear coordinate change?
  - Does I contain a polynomial with at most t terms?
- **Toric ideals are nice**
  - Make interesting varieties but with combinatorial theory.
  - Gröbner bases of toric ideals are polyhedral objects.
  - Toric varieties appear in applications, e.g. discrete exponential families.
- **Any new tool allows new larger scale experiments**.

Examples of hidden toric structure

- Group based models in phylogenetics become toric in Fourier coordinates
- Cumulant coordinates in statistics can dramatically shorten polynomials
  - E.g. the tangential variety to Segre becomes toric (non-linear, though)
- Chemical reaction network theory: toric steady states

Proving that a variety is not (isomorphic to) a toric variety

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Is there an algorithmic (or other) way to prove that a (projective) variety is not isomorphic to a toric variety?

I'd be happy with an algebraic answer (for affine or projective varieties), using the fact that toric ideals are binomial prime ideals. There ne could use that the coordinate rings are characterized as those admitting a fine grading by an affine semigroup , i.e. presented by a binomial prime ideal (Prop. 1.11 in Eisenbud/Sturmfels "Binomial ideals").

This question resulted from an Example that I discussed with Mateusz Michalek. The example is: let V be the Zariski closure of the image of the parameterization:

$$(p_1, p_2, a_1, a_2, a_3, b_1, b_2, b_3) \to \begin{pmatrix} p_1a_1a_2a_3 + p_2b_1b_2b_3 \\ p_1a_1a_2b_3 + p_2b_1b_2a_3 \\ p_1a_1b_2a_3 + p_2b_1a_2b_3 \\ p_1a_1b_2b_3 + p_2b_1a_2a_3 \\ p_1b_1a_2a_3 + p_2a_1b_2b_3 \\ p_1b_1a_2b_3 + p_2a_1b_2a_3 \\ p_1b_1b_2a_3 + p_2a_1a_2b_3 \\ p_1b_1b_2b_3 + p_2a_1a_2a_3 \end{pmatrix}$$

Implicitization using Macaulay2 is quick and yields a complete intersection:

$$\langle et - ry - qu + wo, wt - qy - ru + eo, we - qr - yu + to\rangle \subset k[q, w, e, r, t, y, u, o]$$

How to prove that V is not toric?

toric-varieties ac.commutative-algebra

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asked Apr 5, 2012 at 15:33
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1 Is this toric variety always smooth?

Question feed

No toric structure: Lisa Nicklasson's example

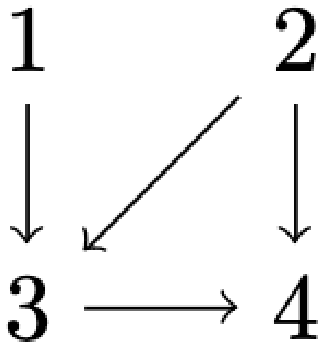
Are all discrete Bayesian networks toric after linear coordinate changes?

Consider the prime ideal generated by

$$\begin{vmatrix} p_{1111}+p_{1112}+p_{1121}+p_{1122} & p_{1211}+p_{1212}+p_{1221}+p_{1222} \\ p_{2111}+p_{2112}+p_{2121}+p_{2122} & p_{2211}+p_{2212}+p_{2221}+p_{2222} \end{vmatrix}$$

and

$$\begin{vmatrix} p_{1111} & p_{2111} \\ p_{1112} & p_{2112} \end{vmatrix}, \begin{vmatrix} p_{1121} & p_{2121} \\ p_{1122} & p_{2122} \end{vmatrix}, \begin{vmatrix} p_{1211} & p_{2211} \\ p_{1212} & p_{2212} \end{vmatrix}, \begin{vmatrix} p_{1221} & p_{2221} \\ p_{1222} & p_{2222} \end{vmatrix}$$



Theorem. No linear coordinate change makes this toric.

No toric structure: Lisa Nicklasson's example

Foundations of Computational Mathematics (2019) 19:1363–1385
https://doi.org/10.1007/s10208-018-9405-0

FOUNDATIONS OF COMPUTATIONAL MATHEMATICS
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When is a Polynomial Ideal Binomial After an Ambient Automorphism?

Lukas Katthän[1] · Mateusz Michałek[2,3] · Ezra Miller[4]

- The locus in $\mathrm{GL}_n(k)$ which makes an ideal binomial is constructible.
- Can in principle be computed using comprehensive Gröbner bases.
- Solves a much more general problem.
- Not efficient enough for above examples.

Goal: Make it efficient in more specific situation.

Approach

A variety is toric if it is the closure of a torus orbit of one point.
Find the maximal torus acting on $V(I)$ in GL_n!

Here *torus* always means algebraic torus, i.e. a commutative group (of matrices) that is isomorphic to $(\mathbb{C}^*)^k$, e.g. a group of diagonal matrices.

Example

- Let $I = \langle xz - y^2 \rangle \in \mathbb{C}[x, y, z]$. Its variety is toric.
- A $2d$-algebraic torus acting on it is generated by the matrices

$$\begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a^{-1} \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b^2 \end{pmatrix}, \quad a, b \in \mathbb{C}^*$$

- $V(xz - y^2)$ is the closure of the orbit of $(1, 1, 1)$ under the action of $(\mathbb{C}^*)^2$.
- The torus is a (diagonal) subgroup of $\mathrm{GL}_3(\mathbb{C})$ and thus easy to find.
- Necessarily this torus must be a subgroup of the subgroup fixing the ideal I!

- Also $V((x + y)z - y^2)$ is a toric variety (obvious?).
- There is some $2d$ (non-diagonal) algebraic torus in the group fixing this ideal.
- Find it! Diagonalize it!

Setup

- Fix $k = \mathbb{C}$, but in reality, start with $\mathbb{Q}$ and algebraically extend as needed.
- $I \subset \mathbb{C}[x_1, ..., x_n]$ is assumed prime and homogeneous.
- The Lie group $\mathrm{GL}_n(\mathbb{C})$ acts on the polynomial ring via

$$g.f(x) = f \circ g^{-1}(x) = f(g^{-1} \cdot x)$$

- Consider the stabilizer of I:

$$G_I := \{g \in \mathrm{GL}_n(\mathbb{C}) \mid g.I \subset I\}$$

  This is itself a Lie group (like any closed subgroup of $\mathrm{GL}_n(\mathbb{C})$).
- Lie algebra $\mathfrak{g}_I$ of G_I can be computed from I using linear algebra.

Symmetry Lie algebras (Maraj and Pal, arXiv:2309.10741)

- The Lie algebra $\mathfrak{g}_I$ can be computed as the stabilizer of an action $*$:
  - $g * c = 0$ for $c \in \mathbb{C}$
  - $g * x_i = g \cdot x_i$
  - $g * (pq) = (g * p)q + p(g * q)$ for all $p, q \in \mathbb{C}[x_1, ..., x_n]$
- Then $\mathfrak{g}_I = \{g \in \mathbb{C}^{n \times n} \mid g * I \subset I\}$
- Question 34: Can symmetry Lie algebras detect if I is toric?

[MP23] criterion: If $\dim(G_I) = \dim(\mathfrak{g}_I) < \dim(I)$ then $V(I)$ is not toric.

This solves Lisa Nicklasson's example again because there $\dim(\mathfrak{g}_I) = 8$ (but 5 equations in 16 variables).

Our plan for Question 34

- If I is prime and $I' = g.I$ is binomial, there is a maximal torus T' in G_I such that the orbit of a generic point in $V(I)$ is dense in $V(I)$.
- Can compute *some* maximal torus T in G_I (or corresponding algebra in $\mathfrak{g}_I$) using Lie theory and Cartan algebras.
- All maximal tori are conjugate, so T and T' are. Thus the orbit of a generic point of $V(I)$ under T is also dense.
- To find the coordinate transform, we diagonalize the torus. Then it acts as

$$((t_1, ..., t_d), (x_1, ..., x_n)) \to (t_1^a x_1, ... t_n^a x_n)$$

for some $a_1, ..., a_n \in \mathbb{Z}^d$.

Computing a maximal torus

- We compute a nilpotent and self-normalizing (Cartan) subalgebra $\mathfrak{c}$.
- Its Lie group is the centralizer of a maximal torus and of the form $T \times C$ with T a maximal torus and C unipotent.
- Any Cartan subgalgebra $\mathfrak{c}$ decomposes as $\mathfrak{t} \oplus \mathfrak{n}$ where $\mathfrak{t}$ consists of diagonalizable and n of nilpotent matrices.
- All elements of $\mathfrak{t}$ commute and are thus simultaneously diagonalizable.
- We can diagonalize the toral subalgebra $\mathfrak{t}$ by Jordan decomposition of a generic element of $\mathfrak{c}$.

The algorithm

1. Compute the Lie algebra $\mathfrak{g}_I$.
2. Pick $x \in \mathfrak{g}_I$ randomly and compute $\mathfrak{c} = \ker\left(\mathrm{ad}(x)^{\dim \mathfrak{g}_I}\right)$.
3. Check if $\mathfrak{c}$ is Cartan, otherwise try again with new x.
4. Decompose $\mathfrak{c} = \mathfrak{t} + \mathfrak{n}$.
5. Diagonalize $\mathfrak{t}$ with a suitable $S \in \mathrm{GL}_n(\mathbb{C})$.
6. Check if $S.I$ is binomial (or toric).

Note: If $V(I)$ is not binomial, then the last step must return FALSE.

The algorithm is implemented in SAGE and runs on relevant examples.

Back to mathoverflow...

```
┌────────────────────────────────────────────────────────────────────┐
│ SageMath version 10.6, Release Date: 2025-03-31                    │
│ Using Python 3.12.5. Type "help()" for help.                       │
└────────────────────────────────────────────────────────────────────┘
sage: load ("isToric.sage")
....: R.<e,t,r,y,q,u,w,o> = PolynomialRing(QQ)
....: I = ideal (e*t-r*y-q*u+w*o, w*t-q*y-r*u+e*o, w*e-q*r-y*u+t*o)
....: ideal_is_toric(I)
[##############################] 100%
searching torus
diagonalizing torus
diagonalizing ideal
checking if ideal is binomial and prime after the coordinate change
[
      [ 1  1  1  1  1  1  1  1]
      [-1  1 -1  1  1 -1  1 -1]
      [ 1  1 -1 -1  1  1 -1 -1]
      [-1  1  1 -1  1 -1 -1  1]
      [-1  1 -1  1 -1  1 -1  1]
      [ 1  1  1  1 -1 -1 -1 -1]
      [-1  1  1 -1 -1  1  1 -1]
True, [ 1  1 -1 -1 -1 -1  1  1]
]
```

Non-homogeneous ideals are in reach

If I is not homogeneous, just test its homogenization I^h!

- We look for affine linear coordinate changes.
- Consider the set of matrices fixing the homogenization variable, i.e.

$$S = \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline \star & & & \\ \vdots & & \star & \\ \star & & & \end{array}\right)$$

- Do all computations in the Lie algebra of $(n+1)$-matrices that have e_0 as a left eigenvector.

Non-reduced schemes are out of reach

- Consider the zero-dimensional(!) binomial ideal $I = \langle x^4, y^4, x^3y - xy^3 \rangle$
- The Lie algebra $\mathfrak{g}_I$ consists only of multiples of the identity.
- Applying a linear coordinate change does not change anything about this.
- Thus we will never find the binomial structure using symmetry Lie algebras.

We just don't understand binomial schemes very well!

Experiments: Toric Gaußian graphical models

- Let G be an undirected graph with edge set E.
- Gaußian graphical model:

$$M_G = \left\{\Sigma \in \mathrm{PD}_n : \Sigma_{ij}^{-1} = 0 \quad \text{for} \quad ij \notin E\right\}$$

- When is (the closure of) M a toric variety?
- Challenge: We don't know many vanishing ideals of M_G.
- Misra/Sullivant: gen. deg. $\leq 2 \Rightarrow$ toric (in original coordinates).

Experiments: Toric Gaußian graphical models

All graphs on 4 vertices:

- Computations in 10 variables.
- All vanishing ideals can be computed.
- Exactly those which are toric in original coordinates are toric.
- Lie algebras are usually large, live in $GL_{10}(\mathbb{C})$ now.

graph	dim model	dim Lie algebra	dim max tori	(can be made) toric
diamond	9	30	6	no
paw	8	52	8	yes
cycle	8	4	4	no
claw	7	37	7	yes
path	7	33	7	yes

Experiments: Toric Gaußian graphical models

5, 6, 7 vertices:

- Computations in 15 / 21 / 28 variables still mostly doable.
- From 6 vertices on, hardly any vanishing ideals known.
- Use CI ideal?
  - This computes a maximal torus acting on one component.
  - But no simple way to compute dimension of orbits or stabilizers.
- No example of an ideal that is toric after coordinate change but not before.

Outlook

- The algorithm always computes a maximal torus acting on $V(I)$.
- Study special cases, e.g. group G_I is a torus, base change is unique.
- Looking for more examples.
- What to do with knowledge about low-codimension tori acting on varieties?
- Code available at **https://github.com/villjulian/isToric**.

Thanks for listening!