

The Geometry of Rank-One Tensor Completion*

Thomas Kahle[†], Kaie Kubjas[‡], Mario Kummer[§], and Zvi Rosen[¶]

Abstract. The geometry of the set of restrictions of rank-one tensors to some of their coordinates is studied. This gives insight into the problem of rank-one completion of partial tensors. Particular emphasis is put on the semialgebraic nature of the problem, which arises for real tensors with constraints on the parameters. The algebraic boundary of the completable region is described for tensors parametrized by probability distributions and where the number of observed entries equals the number of parameters. If the observations are on the diagonal of a tensor of format $d \times \cdots \times d$, the complete semialgebraic description of the completable region is found.

Key words. rank-one tensor, tensor completion, probability distribution, semialgebraic set

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1. Introduction. When approaching high-dimensional tensor data, the large number of entries demands complexity reduction of some sort. One important structure to exploit is sparsity: tensors that have many zero entries can be treated with specialized methods. In this paper we focus on a second concept, separability, which means that tensors have low rank and thus can be parametrized by few parameters. Specifically, we are concerned with the *rank-one completion problem*: Given a subset of the entries of a tensor, does there exist a rank-one tensor whose entries agree with the known data?

Tensor completion is a common task in many areas of science. Examples include compression problems [21] as well as the reconstruction of visual data [23] or telecommunication signals [22]. Tensor completion also appears in the guise of tensor factorization from incomplete data which has many applications and implementations [1]. While we are specifically concerned with the semialgebraic geometry of completability, most of the literature deals with efficiency questions and approximate solutions. The main tool and mathematical hunting

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[†]Otto-von-Guericke Universität, Magdeburg 39106, Germany (thomas.kahle@ovgu.de, <http://www.thomas-kahle.de/>).

[‡]Aalto University, Aalto FI-00076, Finland (kaie.kubjas@aalto.fi, <http://www.kaiekubjas.com>).

[§]Max-Planck-Institute for Mathematics in the Sciences, Leipzig 04103, Germany (kummer@mis.mpg.de, <http://www.math.uni-konstanz.de/~kummer>).

[¶]University of Pennsylvania, Philadelphia, PA 19104 (zvibr@sas.upenn.edu, <https://www.math.upenn.edu/~zvibr/>).

ground there is the minimization of the nuclear norm [10, 23, 6, 34, 33].

Our approach here is directed towards the fundamental mathematical question of a characterization of rank-one completability with a particular emphasis on the real case. What kind of constraints on the entries of a partial tensor guarantee the completability to a rank-one tensor? As recognized in the matrix case, there are combinatorial conditions on the locations of the known entries of a partial tensor. In the best case, which here means working over an algebraically closed field, all additional conditions are algebraic. In all harder and more interesting cases like the completion of real tensors, tensors with linear constraints on the parameters, or even tensors with inequality constraints on the parameters, the answer is almost always semialgebraic, that is, it features inequalities.

The algebraic-combinatorial approach to matrix completion has come up several times in the literature. An original reference is the work of Cohen et al. [7]. A more transparent proof of necessary and sufficient conditions for the existence and uniqueness of a rank-one completion was given in [11]. In this work it became clear that there are combinatorial structures that explain how completability depends on the locations of the observed entries. Algebraic and combinatorial structures underlying the problem were further studied in [16, 15]. The semialgebraic nature of low-rank completion problems is already visible in the matrix case [20]. The present paper continues and extends the results of Kubjas and Rosen. It has been recognized that solving tensor problems exactly is systematically harder than solving matrix problems [13]. In particular, low-rank tensor completion is much more complicated than low-rank matrix completion since tensor rank is much more complicated than matrix rank [17]. The fact that tensor rank depends on the field that one is working with also shines through here (see Example 1.1). Nevertheless, we conceive of our work on the rank-one case as a stepping stone towards the low-rank case.

Our concrete approach is as follows: We consider the parametrization map of rank-one tensors as tensor products of vectors. Restricting this map to a subset of the entries of the tensor, we get parametrizations of partial tensors. Recovery questions can then be asked as questions about the images of the restrictions. Over an algebraically closed field, and with no further restrictions, the image of the parametrization map is quite easy to describe. This is the classical Segre embedding from algebraic geometry. For applications, however, we need to work with constrained sets of tensors and parameters. For example, we may require that whenever the observed entries of a partial tensor are real, the recovery ought to be real, too. We may also choose to restrict parameters to be nonnegative, sum to one, or both. Examples 1.1 and 1.2 show some immediate effects of these requirements. The best possible result in our setup would be a method that allows us to translate arbitrary inequalities and equations in parameter space into a semialgebraic description of the image of restrictions of ϕ . We succeed with such a description in the case of a partial probability tensor (a tensor whose entries form a probability distribution) with given diagonal entries in section 4.

We begin by illustrating the field dependence of tensor completion. A real tensor that has complex tensor rank one also has real tensor rank one, but a similar statement for partial tensors is false. The principal problem is that a complex rank-one tensor can have some real entries so that there exists no real rank-one tensor completing these entries. The following is an adaptation of a standard example on the difference between real and complex tensor rank, going back to Kruskal [18, 19].

Example 1.1. Consider the $2 \times 2 \times 2$ partial tensor with third coordinate slices

$$(1.1) \quad \begin{pmatrix} ? & 1 \\ 1 & ? \end{pmatrix}, \quad \begin{pmatrix} 1 & ? \\ ? & -1 \end{pmatrix},$$

where ? stands for an unspecified entry. Proposition 2.2 shows that the question marks can be filled with complex numbers so that the resulting tensor has rank one. The question marks cannot, however, be filled with real numbers to make a real rank-one tensor. Indeed, if this were the case, then there would be real vectors $\begin{pmatrix} 1 \\ a \end{pmatrix}, \begin{pmatrix} 1 \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \in \mathbb{R}^2$ whose tensor product has the specified entries in (1.1). Here two entries can be chosen to be one by scaling the first two vectors and compensating by the third. In particular, this means that

$$bc = 1, \quad ac = 1, \quad d = 1, \quad abd = -1.$$

It is easy to check that there are only two complex solutions to these equations. In fact, the real rank-one completability of a partial tensor like (1.1) does not depend on the exact values of the specified entries but only their signs. The four entries can be completed to a real rank-one tensor if and only if an even number of them are negative.

The constraints in Example 1.1 are given by equations. The question becomes more interesting with semialgebraic constraints as in the following example.

Example 1.2. Consider real rank-one (2×2) -matrices parametrized as

$$\mathbb{R}^2 \times \mathbb{R}^2 \ni \begin{pmatrix} \theta_{1,1} \\ \theta_{1,2} \end{pmatrix}, \begin{pmatrix} \theta_{2,1} \\ \theta_{2,2} \end{pmatrix} \mapsto \begin{pmatrix} \theta_{1,1}\theta_{2,1} & \theta_{1,1}\theta_{2,2} \\ \theta_{1,2}\theta_{2,1} & \theta_{1,2}\theta_{2,2} \end{pmatrix}.$$

Assume that only the diagonal entries can be observed. It is easy to see that in this case the set of possible diagonal entries is all of \mathbb{R}^2 . In applications in statistics, $\theta_1 = (\theta_{1,1}, \theta_{1,2})$ and $\theta_2 = (\theta_{2,1}, \theta_{2,2})$ may be probability distributions and satisfy $\theta_{1,2} = 1 - \theta_{1,1}$ and $\theta_{2,2} = 1 - \theta_{2,1}$. Not yet imposing nonnegativity, these conditions constrain the diagonal entries x_{11}, x_{22} by

$$(x_{11} - x_{22})^2 - 2(x_{11} + x_{22}) + 1 \geq 0.$$

The yellow region in Figure 1a contains points satisfying this constraint. Introducing inequalities on the parameters θ_1, θ_2 constrains the set of diagonal entries further. For example, if $\theta_{1,1}, \theta_{1,2}, \theta_{2,1}, \theta_{2,2}$ are the entries of probability vectors, they ought to be nonnegative. Figure 1b shows the effect of imposing nonnegativity on $\theta_{1,1}$ and $\theta_{2,1}$ (but not on $1 - \theta_{1,1}$ and $1 - \theta_{2,1}$). According to Theorem 4.6, imposing also the last two conditions leads to the additional inequalities $x_{11} \geq 0, x_{22} \geq 0$, and $x_{11} + x_{22} \leq 1$ (Figure 1c).

In this paper the entries of tensors are indexed by $D = [d_1] \times \cdots \times [d_n]$ where d_1, \dots, d_n are some fixed integers, each larger than one. A *partial tensor* is an array of real or complex numbers indexed by a subset $E \subseteq D$. Field assumptions are important in this work, and we are precise about whether we use \mathbb{R} or \mathbb{C} .

Let \mathbb{F} be either \mathbb{R} or \mathbb{C} . The set of *rank-one tensors* in \mathbb{F}^D is the image of the *parametrization map*

$$(1.2) \quad \mathbb{F}^{d_1} \times \cdots \times \mathbb{F}^{d_n} \rightarrow \mathbb{F}^D, \quad (\theta_1, \dots, \theta_n) \mapsto \theta_1 \otimes \cdots \otimes \theta_n.$$

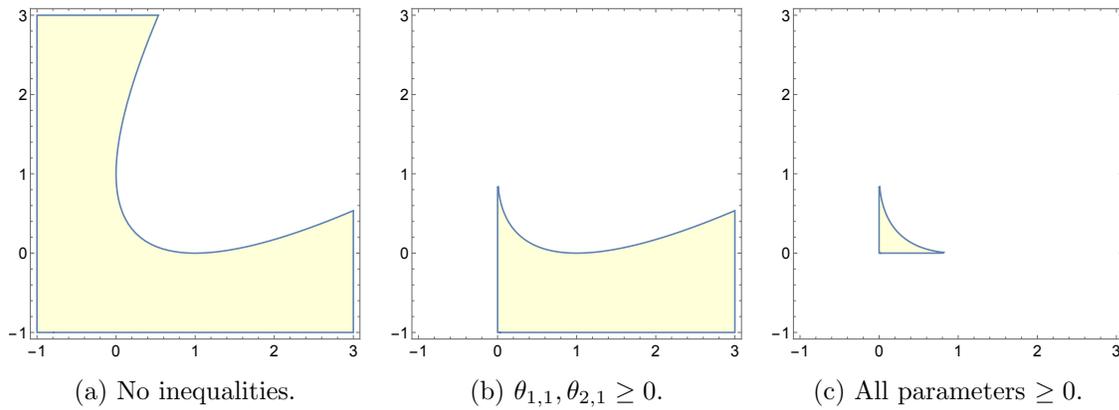


Figure 1. Semialgebraic sets in Example 1.2.

In classical algebraic geometry, the image is known as the Segre variety. It is characterized algebraically by quadratic binomials. To see them, let $N_1 \cup N_2 = [n]$ denote a partition of $[n]$, and $D_i = \prod_{k \in N_i} [d_k]$, $i = 1, 2$. For each partition there is a flattening of a tensor $T \in \mathbb{F}^D$ to a matrix $T \in \mathbb{F}^{D_1 \times D_2}$. A tensor $T \in \mathbb{F}^D$ has rank at most one if and only if all (2×2) -minors of all its flattenings vanish. This gives an explicit set of quadratic equations in the indeterminates $x_i, i \in D$, representing the entries of a tensor. The equations can also be computed by implicitization of the parametrization. Consider the \mathbb{F} -algebra homomorphism

$$(1.3) \quad \psi : \mathbb{F}[x_i : i \in D] \rightarrow \mathbb{F}[\theta_{j,k} : j \in [n], k \in [d_j]], \quad x_{i_1, \dots, i_n} \mapsto \prod_{j=1}^n \theta_{j, i_j}.$$

The toric ideal defining the Segre variety equals $\ker(\psi)$. As always in toric algebra, ψ corresponds to a linear map

$$(1.4) \quad \mathbb{Z}^D \rightarrow \mathbb{Z}^{d_1 + \dots + d_n}, \quad e_{i_1, \dots, i_n} \mapsto \sum_{j=1}^n e_{j, i_j},$$

whose matrix in the standard basis we denote by $A \in \{0, 1\}^{(d_1 + \dots + d_n) \times D}$. See [30, Chapter 4] for an introduction to toric algebra. For any subset $E \subseteq D$, let A_E be the matrix whose columns are exactly the columns of A with indices in E . The toric ideal I_E corresponding to A_E equals the elimination ideal $\ker(\psi) \cap \mathbb{F}[x_i : i \in E]$. In general, equations for I_E are not easy to determine. They could be easily read off a universal Gröbner basis of I_D , but not much is known about this for general n , even in the binary case $d_1 = \dots = d_n = 2$. Isaac Burke has started to classify elements of the universal Gröbner bases for binary rank-one tensors but has encountered very intricate combinatorial structures. More information about this will appear in his forthcoming Ph.D. thesis. If desired, the question of computing a universal Gröbner basis can be formulated as a combinatorial question about hypergraphs [26, Corollary 2.11], but one should not hope that the complications miraculously disappear in this perspective.

In the remainder of the introduction we outline our specific results together with concrete applications. In section 2 we study the existence and finiteness of a rank-one completion of

a partial tensor when no further equations and inequalities are assumed. We investigate the existence of completions over \mathbb{C} and \mathbb{R} and in particular in the presence of zeros. Finitely completable entries are characterized using matroid theory in Proposition 2.8. Unique completability of a partial tensor is investigated in Corollary 2.10.

Lower bounds on the number of observed entries for perfect recovery of a low-rank matrix or tensor have been studied in [4, 5, 24, 35], to name only a few references. An important assumption in these papers is that the positions of observed entries are sampled uniformly randomly. Proposition 2.8 and Corollary 2.10 characterize when a partial tensor is finitely and uniquely completable to a rank-one tensor without any assumptions on the sampling of the entries. This extends the unique and finite completability in the matrix case [11, 28, 16] to tensors. These results can also be used to design small sets of locations of observed entries that guarantee finite or unique completability to a rank-one tensor. In the algebraic geometry language above, tensor completion is equivalent to solving a system of polynomial equations. Thus, Gröbner basis methods and numerical algebraic geometry give effective methods for rank-one tensor completion in the noiseless case, again with no assumptions on the sampling of the entries.

Results in section 2 also have possible applications in chemometrics. In [3], Appellof and Davidson apply tensor decompositions of third order tensors to analysis of multicomponent fluorescent mixtures. There, rank-one tensors correspond to solutions with only one fluorophore. In practice, some of the measurements in a chemometry will not reach the excitation level and are thus missing [2]. Using our results in section 2, one can verify whether tensors with missing values correspond to solutions with only one fluorophore. However, since real-world data has noise, in this and other applications a further procedure to measure the distance from a (semi)algebraic set would be necessary.

In section 3 and in particular Theorem 3.13, we find a description of the algebraic boundary of the completable region in the case that all parameters belong to the probability simplex and the number of observations equals the number of free parameters. This is an important step towards deriving a semialgebraic description of the completable region. Finally, section 4 illustrates in the diagonal case that it is possible to achieve totally effective semialgebraic descriptions.

2. Algebraic and combinatorial conditions for completability. Throughout this section, let $E \subseteq D$ denote a fixed index set. A *partial tensor* is an element $T_E \in \mathbb{F}^E$. Here the subscript E serves as a reminder that the tensor is partial. For each full tensor $T \in \mathbb{F}^D$, the *restriction* $T|_E \in \mathbb{F}^E$ to E consists of only those entries of T indexed by E . If a tensor $T = \theta_1 \otimes \cdots \otimes \theta_n$ is of rank one, then any zero coordinate in one of the θ_i yields an entire slice of zeros in T . A first condition on completability of a partial tensor results from this combinatorics of zeros.

Definition 2.1. Fix $j \in [n]$ and $i_j \in [d_j]$. A maximal slice of a partial tensor $T \in \mathbb{R}^E$ is the tensor with index set $E \cap [d_1] \times \cdots \times [d_{j-1}] \times \{i_j\} \times [d_{j+1}] \times \cdots \times [d_n]$ which arises from T by fixing the j th index as i_j . A partial tensor is zero-consistent if every zero entry is contained in a maximal dimensional slice consisting of only zero entries.

The following proposition uses elimination so that \mathbb{F} needs to be algebraically closed.

Proposition 2.2. A partial tensor $T_E \in \mathbb{C}^E$ equals the restriction of a rank-one tensor

$T \in \mathbb{C}^D$ to E if and only if the following two conditions hold:

1. The partial tensor T_E is zero-consistent.
2. The variety $V(I_E)$ contains T_E .

Proof. If $T \in \mathbb{C}^D$ is of rank one, then $T = \theta_1 \otimes \cdots \otimes \theta_n$ for some vectors $\theta_i \in \mathbb{C}^{d_i}$. Therefore, itself and any restriction to an index set $E \subseteq D$ are zero-consistent. It is also clear that it lies in $V(I_D)$, and since $V(I_E)$ is the closure of the projection of $V(I_D)$ it contains $T|_E$.

Now let $T_E \in \mathbb{C}^E$ be a zero-consistent partial tensor contained in $V(I_E)$. Without loss of generality, we can assume that T_E has no zero entry. Indeed, from a partial tensor with consistent zeros we can drop the zero-slices from the notation, complete, and then insert appropriate zero-slices into the completion.

For any $E' \subseteq D$, $I_{E'}$ is a toric ideal and has a universal Gröbner basis consisting only of binomials $x^u - x^v$ such that x^u and x^v are not divisible by a common variable. Said differently, when considered in a specific variable x_i , these binomials are of the form $gx_i^n + h$ where g, h are monomials that do not use the variable x_i . The extension theorem [8, Theorem 3.1.3] states that outside the vanishing locus of the polynomials g , a point of $V(I_{E' \setminus \{i\}})$ can be extended to a point of $V(I_{E'})$. Since all g are monomials, their vanishing locus is contained in the coordinate hyperplanes. Hence every partial tensor $T_E \in V(I_E)$ without zero entries can be completed to $T \in V(I_D)$ by applying the extension theorem $|D \setminus E|$ times. ■

Remark 2.3. The second condition in Proposition 2.2 need not necessarily be checked on the toric ideal I_E , which may be computationally unavailable. There are several binomial ideals that have I_E as their radical and thus define the same variety. A natural example is the circuit ideal C_E generated by all binomials corresponding to circuits of A_E . By [9, Proposition 8.7] it holds that $V(C_E) = V(I_E)$.

Example 2.4. Example 1.1 shows that Proposition 2.2 fails if \mathbb{C} is replaced by \mathbb{R} . For $E = \{112, 121, 211, 222\}$ the restricted matrix A_E has full rank and thus $V_{\mathbb{R}}(I_E) = \mathbb{R}^4$ while the given T_E has no real rank-one completion.

Remark 2.5. Rank-one matrix completion can be studied combinatorially using graph theory. Given a partial matrix $T_E \in \mathbb{R}^E$, one can define a bipartite graph G with vertex set $[d_1] \times [d_2]$ and edge set E . The rank-one matrix completions are studied using zero-entries and cycles of G . Zero-consistency of a matrix is called *singularity with respect to 3-lines* in [7] and *the zero row or column property* in [11]. A partial matrix T_E satisfies the second condition in Proposition 2.2 if and only if on every cycle C of G , the product over the edges with even indices equals the product over the edges with odd indices. This observation follows from the explicit description of the generators of the universal Gröbner basis in terms of cycles on the bipartite graph [32, Proposition 4.2]. In [25] the Graver basis is computed, which in this case coincides with the universal Gröbner basis. This condition is called *singularity with respect to cycles* in [7] and *the cycle property* in [11]. Rank-one matrices have a square-free universal Gröbner basis. Therefore all uses of the extension theorem as in Proposition 2.2 yield linear equations. In the tensor case the combinatorial interpretations break down and, for example, the universal Gröbner basis of rank-one $2 \times 2 \times 2$ tensors is of degree three and not square-free. However, iterative algorithms using the extension theorem or related methods also work for tensors.

The problem of real rank-one completion is, for each $E \subseteq D$, to determine the difference of the image of real and complex rank-one tensors T under the restriction map $T \mapsto T|_E$. This problem leads to a combinatorial problem in \mathbb{Z} -linear algebra clarified in Proposition 2.7. Consider a partial tensor $T_E \in \mathbb{R}^E$ that satisfies the conditions in Proposition 2.2. As in the proof of Proposition 2.2 we can assume that T_E has no zero entries as these would be contained in zero-slices which we can ignore for completion. The parametrization (1.2) of the entries $T_e, e \in E$ of a completion of T_E takes the form

$$(2.1) \quad T_e = \theta_{1,e_1} \cdots \theta_{n,e_n}, \quad e \in E,$$

where θ_{j,e_j} denotes the e_j th component of θ_j . Completability questions are questions about the solutions of this system of binomial equations. Additionally assume that E meets every maximal dimensional slice of D , which implies that every parameter $\theta_{j,k}$ appears at least once in (2.1). Given $T_e \neq 0, e \in E$, this implies that any solution has only nonzero values for the parameters. This means that the ideal generated by (2.1) can be considered as an ideal in Laurent polynomial ring $\mathbb{F}[\theta_{j,k}^\pm, j = 1, \dots, n, k \in [d_j]]$, and the theory of [9, section 2] applies. In particular, the equations can be diagonalized by computing the Smith normal form of the exponent vectors of the monomials appearing in (2.1), which corresponds to a multiplicative coordinate change.

Example 2.6. The equations in Example 1.1 can be diagonalized to

$$x^2 = -1, \quad y = 1, \quad z = 1, \quad w = 1,$$

where $x^2 = ab(bc)^{-1}(ac)^{-1} = c^{-2}$, $y = ac$, $z = bc$, $w = d$.

As a consequence of the diagonalization argument that gives [9, Theorem 2.1(b)], we get the following proposition.

Proposition 2.7. *For a given subset $E \subseteq D$ the following are equivalent:*

- (i) *Every real partial tensor $T_E \in \mathbb{R}^E$ with nonzero entries which is completable over the complex numbers is also completable over the real numbers.*
- (ii) *The index of the lattice spanned by the columns of A_E in its saturation is odd.*

Moreover, whether a real partial tensor $T_E \in \mathbb{R}^E$ with nonzero entries which is completable over the complex numbers is also completable over the real numbers depends only on the signs of the entries of T_E .

Proof. Since we assume the entries to be nonzero (i) is equivalent to the homomorphism of tori $\psi : (\mathbb{C}^*)^r \rightarrow (\mathbb{C}^*)^s$ (for suitable r and s) corresponding to our parametrization having the property that every real point in the image has a real point in its preimage. After applying suitable group automorphisms of $(\mathbb{C}^*)^r$ and $(\mathbb{C}^*)^s$ we can assume that ψ is of the form $\psi(x_1, \dots, x_r) = (x_1^{a_1}, \dots, x_r^{a_r}, 1, \dots, 1)$ if $s \geq r$ and $\psi(x_1, \dots, x_r) = (x_1^{a_1}, \dots, x_s^{a_s})$ otherwise (this corresponds to computation of the Smith normal form of A_E). Group automorphisms of $(\mathbb{C}^*)^r$ send real points to real points. Thus, (i) holds if and only if all a_i are odd. Since the index of the lattice spanned by the columns of A_E in its saturation is the product of the a_i the claim follows.

For the second part assume that a_1, \dots, a_k are odd and a_{k+1}, \dots, a_r are even. Consider first the case that $s \geq r$. Then T_E is completable over the real numbers if and only if it is in

the preimage of

$$(\mathbb{R}^*)^k \times (\mathbb{R}_{>0})^{r-k} \times \{1\}^{s-r}$$

under the above automorphism of $(\mathbb{C}^*)^s$. Assuming completability over the complex numbers, this translates to conditions on the signs of the entries of T_E . A similar argument applies for $s < r$. ■

In Proposition 2.7, the assumption that the partial tensor T_E has nonzero entries is no loss of generality, since any zero entry of a rank-one tensor is contained in a maximal dimensional slice of zeros. These maximal dimensional slices of zeros originate from parameters being zero and can be dealt with separately.

Given completability over a fixed field, one can ask about uniqueness properties of the completion. More generally, for some of the completed entries there could be only finitely many choices while others can vary continuously. An entry that has only finitely many possible values for rank-one completion, is called *finitely completable*. An entry that has only one possible value for rank-one completion is called *uniquely completable*. The occurrence of finitely completable entries is natural. For example, if three entries of a rank-one (2×2) matrix are given, the determinant becomes a linear polynomial determining the fourth entry. The proof of Proposition 2.2 shows how the finitely completable entries are solutions of certain binomial equations. In this context, an important observation is that, generically, the locations of the finitely completable entries do not depend on the entries of T_E but just the combinatorics of E . This statement for matrices can be found in [16, Theorem 10].

Proposition 2.8. *There is a matroid on ground set D with closure function $\text{cl} : 2^D \rightarrow 2^D$ having the following property: Let $E \subseteq D$ be any index set with $T_E \in \mathbb{C}^E$ a generic partial tensor completable according to Proposition 2.2. Then the closure $\text{cl}(E)$ consists exactly of the entries that are finitely completable from the entries in E .*

Proof. This follows from Proposition 2.9 below. ■

The gist of Proposition 2.8 is that for generic T_E , the set of finitely completable entries does not depend on the entries of T_E but only on E . Even more, $\text{cl}(E)$ is an honest closure relation on explicit matroids. The following matroids can be used:

- the column matroid of the Jacobian of the parametrization (1.2).
- the algebraic matroid of the toric ideal $\ker(\psi)$ in (1.3).
- the column matroid of the matrix A defining (1.4).

The equivalence of these three matroids is well known. The algebraic matroid of the coordinate ring of a toric ideal equals the linear matroid of the defining matrix. The equivalence of the first and second matroid follows from [14, Proposition 2.14]. The closure function can be specified algebraically as follows. For any index set $E \subseteq D$, let $\mathbb{C}[E] := \mathbb{C}[x_e : e \in E]$ be a polynomial ring with one indeterminate for each entry of a partial tensor with index set E .

Proposition 2.9. *The closure function $\text{cl} : 2^D \rightarrow 2^D$ of the algebraic matroid defined by the ideal I is the function which maps a set $E \subseteq D$ to the largest set E' containing E such that the projection $\mathbb{C}^{E'} \rightarrow \mathbb{C}^E$ induces a generically finite-to-one map on $V(I_D \cap \mathbb{C}[E']) \rightarrow V(I_D \cap \mathbb{C}[E])$.*

Proof. The closure $\text{cl}(E)$ of E consists of all elements dependent on E . For an algebraic matroid, $\text{cl}(E)$ consists of all elements algebraic over E . Given a point in $V(I_D \cap \mathbb{C}[E])$ and

$f \in \text{cl}(E) \setminus E$, substituting the indeterminates $x_e, e \in E$ with its coordinates, the algebraic dependence of f on E yields a univariate polynomial in x_f . As for the elements not in the closure of E , since they are not algebraic over E , they can take infinitely many values; in particular, the fiber over the generic point in that projection is infinite. ■

Proposition 2.7 also gives a characterization when a partial tensor is uniquely completable to a rank-one tensor. Also here the assumption on nonzero entries can be dealt with separately.

Corollary 2.10. (i) *A partial tensor with nonzero entries is uniquely completable to a complex rank-one tensor if and only if it is finitely completable and the lattice spanned by the columns of A_E is saturated.*

(ii) *A real partial tensor with nonzero entries is uniquely completable to a real rank-one tensor if and only if it is finitely completable and the index of the lattice spanned by the columns of A_E in its saturation is odd.*

Proof. As in the proof of Proposition 2.7, we can assume that ψ is of the form $\psi(x_1, \dots, x_r) = (x_1^{a_1}, \dots, x_r^{a_r}, 1, \dots, 1)$. A point in the image has a unique complex preimage if and only if a_1, \dots, a_r are all one. A real point in the image has a unique real preimage if and only if a_1, \dots, a_r are all odd. ■

Example 2.11. In the matrix case, the finitely completable entries of a generic partial matrix form a block diagonal partial matrix after a suitable indexing of rows and columns (we do not assume that matrices or blocks are square matrices). The reason is that the closure operation on the Jacobian matroid can be interpreted as the closure operation on the graphic matroid of the bipartite graph whose vertices are the row and column labels and whose edges correspond to E . This closure completes connected components to complete bipartite graphs, and a bipartite graph where all connected components are complete corresponds to a block diagonal matrix after a suitable indexing of rows and columns. An analogous statement for tensors is not true. For example, the partial $2 \times 2 \times 2$ tensor with observed entries at positions $(1, 1, 2), (1, 2, 1), (2, 1, 1)$ (the blue entries in Figure 2) has no finitely completable entries, and it cannot be transformed to block diagonal form after a suitable indexing.

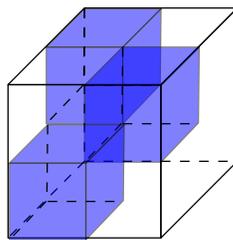


Figure 2. $2 \times 2 \times 2$ tensor with observed entries $(1, 1, 2), (1, 2, 1), (2, 1, 1)$.

3. The algebraic boundary of the completable region. The *algebraic boundary* of a semialgebraic set $S \subseteq \mathbb{R}^n$ is the Zariski closure of the boundary of S in the Euclidean topology.

Theorem 3.13 is a description of the algebraic boundary of the completable region in the case that the parameters form probability distributions and the number of observations equals the number of free parameters. A result of Sinn gives that the algebraic boundary is defined by a single polynomial (Proposition 3.3) which we show to be a product of a special irreducible polynomial with indeterminates. To this end we study the Jacobian of the parametrization map and the factorization of its determinant. We compute the locus where the Jacobian has rank deficit (Propositions 3.8 and 3.12), and then argue about the relation of this set to the algebraic boundary.

The set $E \subseteq D$ again denotes the index set of observed entries of a tensor. Consider the restricted parametrization

$$p : \Delta^{d_1-1} \times \dots \times \Delta^{d_n-1} \rightarrow \mathbb{R}^E,$$

where Δ^{m-1} is the $(m - 1)$ -dimensional simplex that is the convex hull of the unit vectors in \mathbb{R}^m . We write $N = \{(j, k) : j \in [n], k \in [d_j]\}$ for the set indexing dimensions of the general parametrization (1.2) of rank-one tensors. In the above parametrization the parameters are linearly dependent and this dependence, together with nonnegativity, leads to semialgebraic constraints on the image of p , the *completable region*. Write $\tilde{N} = \{(j, k) : j \in [n], k \in [d_j - 1]\}$ for the index set of linearly independent parameters. From now on, let $E \subseteq D$ be of size $\sum_{i=1}^n (d_i - 1)$. With this requirement, we set the number of observations equal to the dimension of the parameter space.

The vanishing ideal of the graph of the parametrization is $G_E = \langle x_i - p_i : i \in E \rangle$. By assumption, the Jacobian matrix J_E of p is a square matrix. We also assume that the completable region has nonempty interior and that for each pair $(j, k) \in N$ there is at least one element in E that has k at its j th position. If it has empty interior, then its algebraic boundary is just its Zariski closure, which can be determined by eliminating the parameters from the vanishing ideal G_E of the graph of p . This happens, for example, when the number of observations exceeds the dimension of the parameter space, that is, $|E| > \sum_{i=1}^n (d_i - 1)$. The assumption on the index set E is necessary so that the map p captures information about each parameter. It is satisfied exactly if E meets every maximal dimensional slice of D . For example, for matrices this means that E meets each row and each column.

First we show that the algebraic boundary of the completable region is defined by a single nonzero polynomial. The following lemma is a version of the generic smoothness lemma. In its proof we use the same notation as [12].

Lemma 3.1. *Let $g : \mathbb{R}^k \rightarrow \mathbb{R}^k$ be a polynomial map whose image has nonempty interior. The Jacobian determinant is not identically zero.*

Proof. Let $h : \mathbb{A}^k \rightarrow \mathbb{A}^k$ be the morphism of affine varieties given by the same polynomials as g . By assumption h is dominant. Thus, by [12, Lemma III.10.5] there is a nonempty Zariski open subset $U \subseteq \mathbb{A}^k$ such that $h|_U$ is smooth (of relative dimension zero). Thus, by Proposition [12, Proposition III.10.4] the sheaf of relative differentials of U over \mathbb{A}^k is locally free, which means that the Jacobian matrix is invertible at all points of U . ■

Lemma 3.2. *Let $g : \mathbb{R}^k \rightarrow \mathbb{R}^k$ be a polynomial map. Let $S \subseteq \mathbb{R}^k$ be a semialgebraic set contained in the closure of its interior in the Euclidean topology. If the image $g(S)$ of S has nonempty interior, then $g(S)$ is contained in the closure of its interior in the Euclidean*

topology.

Proof. Let J be the Jacobian matrix of g . Since $g(S)$ has nonempty interior, the Jacobian determinant $\det(J)$ is not identically zero by Lemma 3.1. Since $D = \{x \in \mathbb{R}^k : \det(J(x)) = 0\}$ contains no nonempty open set and by the assumption on S , the closure of $S' = \text{int}(S) \setminus D$ in the Euclidean topology contains S . It follows that the closure of $g(S')$ in the Euclidean topology contains $g(S)$. The inverse function theorem implies that $g(S')$ is contained in the interior of $g(S)$. Thus, the claim follows. ■

Proposition 3.3. *The algebraic boundary of the completable region is of pure codimension one, that is, it is the zero set of a nonzero polynomial.*

Proof. By [29, Lemma 4.2], if a semialgebraic set $S \subset \mathbb{R}^k$ is nonempty and contained in the closure of its interior in the Euclidean topology and the same is true for its complement $\mathbb{R}^k \setminus S$, then the algebraic boundary of S is a variety of pure codimension one. We will show that these assumptions hold for the image of p and its complement. The image of p is clearly nonempty. It is contained in the closure of its interior in the Euclidean topology by Lemma 3.2. Also, the image of p is clearly not all of \mathbb{R}^m since each coordinate takes a value between 0 and 1, and it is closed in the Euclidean topology since p is continuous and maps from a compact space. Thus, the complement of the image of p is nonempty and open in the Euclidean topology. Therefore, the assumptions of [29, Lemma 4.2] are satisfied and the claim follows. ■

3.1. The Jacobian determinant of the parametrization. In order to find the polynomial that defines the algebraic boundary of the completable region we compute the determinant of the Jacobian J_E (Theorem 3.13). The following example illustrates the results in the next two subsections.

Example 3.4. Assume the indices of observed entries of a $2 \times 2 \times 2$ tensor are $(2, 1, 1)$, $(1, 2, 1)$, and $(1, 1, 2)$. Denote $l_i = 1 - \theta_i$ for $i = 1, 2, 3$. Define the ideal

$$G_E = \langle x_{211} - l_1\theta_2\theta_3, x_{121} - \theta_1l_2\theta_3, x_{112} - \theta_1\theta_2l_3 \rangle.$$

The Jacobian matrix of the parametrization map equals

$$J_E = \begin{pmatrix} -\theta_2\theta_3 & l_1\theta_3 & l_1\theta_2 \\ l_2\theta_3 & -\theta_1\theta_3 & \theta_1l_2 \\ \theta_2l_3 & \theta_1l_3 & -\theta_1\theta_2 \end{pmatrix}$$

and has determinant

$$\theta_1^2\theta_2\theta_3 + \theta_1\theta_2^2\theta_3 + \theta_1\theta_2\theta_3^2 - 2\theta_1\theta_2\theta_3 = \theta_1\theta_2\theta_3(-\theta_1 - \theta_2 - \theta_3 + 2).$$

The product of polynomials of the parametrization map is $\theta_1^2\theta_2^2\theta_3^2l_1l_2l_3$. Division by $\theta_1\theta_2\theta_3l_1l_2l_3$ yields the monomial $m = \theta_1\theta_2\theta_3$ which divides the determinant as claimed by Proposition 3.8. Consider the matrix

$$B_E = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}.$$

The construction of this matrix is explained in subsection 3.2. The kernel of B_E is spanned by $v = (-1, -1, -1, 2)^T$. Let $l_E = -\theta_1 - \theta_2 - \theta_3 + 2$ be the linear polynomial whose coefficients are equal to the entries of v , as suggested by Proposition 3.12. The Jacobian determinant equals l_E multiplied with m .

Compared to the general parametrization (1.2), the map p has linear restrictions on the coordinates of its domain. We prove our results for a slight generalization of this. To this end, let $\theta_{j,k}, (j, k) \in \tilde{N}$ be indeterminates. For any $(j, k) \in N$, let $l_{j,k}$ be a linear polynomial in the indeterminates $\theta_{j,1}, \dots, \theta_{j,d_j-1}$. In our context $l_{j,k} = \theta_{j,k}$ for $(j, k) \in \tilde{N}$ and $l_{j,d_j} = 1 - \sum_{k=1}^{d_j-1} \theta_{j,k}$ otherwise. In this representation, the parametrization takes the form

$$(3.1) \quad p_{i_1, \dots, i_n} = \prod_{j=1}^n l_{j, i_j}.$$

We prove the following results for this generalized parametrization. An entry of the Jacobian matrix is indexed by a pair $((i_1, \dots, i_n), (j, k))$ of indices $(i_1, \dots, i_n) \in E$, $(j, k) \in N$ and is given by

$$\frac{\partial}{\partial \theta_{j,k}} \prod_{m=1}^n l_{m, i_m} = \frac{\partial l_{j, i_j}}{\partial \theta_{j,k}} \prod_{\substack{m=1 \\ m \neq j}}^n l_{m, i_m}.$$

In the Leibniz determinant formula, each summand is a product

$$(3.2) \quad \sum_{\sigma \in S_{|E|}} \text{sgn}(\sigma) \prod_{(j,k) \in N} \frac{\partial l_{j, \sigma(i)_j}}{\partial \theta_{j,k}} \prod_{\substack{m=1 \\ m \neq j}}^n l_{m, \sigma(i)_m}.$$

For $j \in [n]$ and $k \in [d_j]$, let $\alpha(j, k) = |\{i \in E : i_j = k\}|$ denote the number of times the linear polynomials $l_{j,k}$ is used in the parametrizations p_i for $i \in E$. Each term, and thus the entire determinant in (3.2), is divisible by the product of linear polynomials

$$(3.3) \quad \prod_{(j,k) \in N} l_{j,k}^{\alpha(j,k)-1}.$$

Consider the polynomial l_E arising upon division of the Jacobian determinant by the product (3.3). We show a degree bound on the determinant which yields that l_E is either zero or a polynomial of degree at most one. We use some technical lemmata, the first of which is inspired by [27, Lemma 4.7].

Lemma 3.5. *Let \mathbb{F} be a field. Let M be an $n \times n$ matrix with entries in $\mathbb{F}[x_1, \dots, x_m]$. If an irreducible polynomial $f \in \mathbb{F}[x_1, \dots, x_m]$ divides every $(r + 1)$ -minor of M , then f^{n-r} divides $\det(M)$.*

Proof. Without loss of generality, we can assume $\det(M) \neq 0$. The proof is by induction on n . For $n = 1$ the statement is trivial. If n is arbitrary and $r \geq n - 1$, then the assumption and the conclusion are the same. If $r < n - 1$, the induction hypothesis yields that f^{n-r-1}

divides every $(n - 1)$ -minor of M . Therefore, the adjugate matrix $\text{adj}(M)$ can be factored as $\text{adj}(M) = f^{n-r-1}M'$ for some $n \times n$ matrix M' with entries in $\mathbb{F}[x_1, \dots, x_m]$. It follows that

$$\det(M)^{n-2}M = \text{adj}(\text{adj}(M)) = \text{adj}(f^{n-r-1}M') = f^{(n-r-1)(n-1)}\text{adj}(M').$$

If f divides every entry of M , then f^n divides $\det(M)$. If it does not, then $f^{(n-r-1)(n-1)}$ divides $\det(M)^{n-2}$. This implies that f^{n-r} divides $\det(M)$: If s is the power with which f appears in the factorization of $\det(M)$, then $s \leq n - r - 1$ implies $s(n - 2) \leq (n - r - 1)(n - 2) < (n - r - 1)(n - 1)$. ■

Lemma 3.6. *Let \mathbb{F} be a field and M an $n \times n$ matrix whose entries $M_{ij} \in \mathbb{F}[x_1, \dots, x_m]$ are not necessarily homogeneous polynomials of degree at most d . Let $\text{in}(M)$ be the matrix whose (i, j) th entry is the standard graded leading form of the (i, j) th entry of M . Let Q be the quotient field of $\mathbb{F}[x_1, \dots, x_m]$, and let r be the dimension of the kernel of $\text{in}(M)$ considered as a Q -linear map. Then $\deg(\det(M)) \leq nd - r$.*

Proof. Let M^h denote the matrix whose entries are the homogenizations of the entries of M by a new indeterminate x_0 so that $\text{in}(M) = M^h_{|x_0=0}$. We show that $\det(M^h)$ is divisible by x_0^r . By assumption, all $(n - r + 1)$ -minors of $\text{in}(M)$ vanish, and thus each $(n - r + 1)$ -minor of M^h is divisible by x_0 . It follows from Lemma 3.5 that $\det(M^h)$ is divisible by x_0^r . Since the degree of $\det(M^h)$ is at most nd and since $M = M^h_{|x_0=1}$, the claim follows. ■

Lemma 3.7. *For any $E \subseteq D$ with $|E| = |\tilde{N}|$, the Jacobian determinant (3.2) has degree at most $(|\tilde{N}| - 1)(n - 1)$.*

Proof. Let $\text{in}(p_i)$ be the leading form of p_i . If $\theta_{j,k}$ appears in p_i , then it appears in $\text{in}(p_i)$ with degree one. Hence the matrix of leading forms of the Jacobian is $\text{in}(J) = \left(\frac{\partial \text{in}(p_i)}{\partial \theta_{j,k}}\right)_{i,(j,k)}$. By Lemma 3.6 it suffices to exhibit $(n - 1)$ linearly independent relations among the columns of $\text{in}(J)$. As visible from (3.1), for each $j \in [n]$, the polynomial $\text{in}(p_i)$ is a homogeneous function in the subset $\Theta_j = \{\theta_{j,k}, k \in [d_j]\}$ of the variables. Therefore, according to Euler's theorem on homogeneous functions,

$$\sum_{k=1}^{d_j} \theta_{j,k} \frac{\partial \text{in}(p_i)}{\partial \theta_{j,k}} = \text{in}(p_i), \quad j \in [n], i \in E.$$

Equating these for different j yields $n - 1$ relations among the columns of $\text{in}(J)$

$$\sum_{k=1}^{d_j} \theta_{j,k} \text{in}(J)_{i,(j,k)} = \sum_{k=1}^{d_{j'}} \theta_{j',k} \text{in}(J)_{i,(j',k)}, \quad j, j' \in [n], i \in E.$$

This completes the proof. ■

The degree bound in Lemma 3.7 together with the divisibility by the product (3.3) implies the following form of the Jacobian determinant.

Proposition 3.8. *The determinant of the Jacobian equals*

$$l_E \prod_{(j,k) \in N} l_{j,k}^{\alpha(j,k)-1},$$

where $l_E \in \mathbb{R}[\theta_{j,k}, (j, k) \in D]$ is of degree at most one.

Proof. By Lemma 3.7 the degree of l_E is bounded from above by

$$(|\tilde{N}| - 1)(n - 1) - \sum_{(j,k) \in N} (\alpha(j, k) - 1) = (|\tilde{N}| - 1)(n - 1) - n|E| + |N|.$$

Since $|E| = |\tilde{N}| = |N| - n$, this bound equals

$$(|E| - 1)(n - 1) - n|E| + |E| + n = 1. \quad \blacksquare$$

3.2. A linear factor of the Jacobian determinant. To determine the linear form l_E in Proposition 3.8 we restrict back to the most relevant case in which the linear forms $l_{(j,k)}$ are equal to $\theta_{(j,k)}$ whenever $(j, k) \in \tilde{N}$ and $1 - \sum_k \theta_{(j,k)}$ otherwise. Consider the matrix A defining the linear map (1.4). From A , extract the submatrix \tilde{B}_E consisting of the rows corresponding to indices $(j, k) \in \tilde{N}$ and the columns corresponding to indices in E , and let \tilde{B}_E^T be its transpose. For the entries $b_{i,(j,k)}$ of \tilde{B}_E^T this means that $b_{i,(j,k)} = 1$ if i has k at its j th position and $b_{i,(j,k)} = 0$ otherwise. In other words, the $(i, (j, k))$ entry of \tilde{B}_E^T is 1 if and only if the parameter corresponding to (j, k) appears in p_i . Let

$$B_E = \begin{pmatrix} & 1 \\ \tilde{B}_E^T & \vdots \\ & 1 \end{pmatrix}.$$

Under the assumption that the completable region has nonempty interior, B_E has full rank. Indeed, its transpose defines the toric ideal I_E which is zero by the assumption. In this section we show how the kernel of B_E determines the linear polynomial l_E . To this end, consider the matrix \tilde{J}_E that arises from J_E by dividing the i th row by p_i for each $i \in E$ and multiplying the column (j, k) by $\theta_{k,l}$ for each $(j, k) \in \tilde{N}$. The following lemma is immediate from Proposition 3.8.

Lemma 3.9. *The determinant of \tilde{J}_E equals*

$$\frac{l_E}{\prod_{j=1}^n (1 - (\theta_{j,1} + \dots + \theta_{j,d_j-1}))}.$$

Lemma 3.10. *The determinant of \tilde{B}_E^T is the constant term of l_E .*

Proof. The matrix \tilde{B}_E^T arises from \tilde{J}_E by evaluating all indeterminates $\theta_{(j,k)}$ at zero. By Lemma 3.9 its determinant is the constant of l_E . ■

Lemma 3.11. *The coefficient of $\theta_{(j,k)}$ in l_E is the determinant of the matrix arising from \tilde{B}_E^T after replacement of the (j, k) -column with the all -1 vector.*

Proof. Let C_1 be the matrix obtained from multiplying the (j, k) -column of \tilde{J}_E with $(1 - (\theta_{j,1} + \dots + \theta_{j,d_j-1}))$. Let C_2 be the matrix obtained from C_1 after evaluating all parameters except for $\theta_{j,k}$ at zero. By construction, the determinant of C_2 is $a + b\theta_{j,k}$ where a is the

constant term of l_E and b is the coefficient of $\theta_{j,k}$ in l_E . Thus, the matrix C_3 obtained from C_2 by evaluating $\theta_{j,k}$ at 1 has determinant $a+b$. By construction, the i th entry of the (j,k) th column of C_3 is -1 if $(1 - (\theta_{j,1} + \cdots + \theta_{j,d_j-1}))$ appears in p_i and zero otherwise. All other columns of C_3 are equal to the respective columns of \tilde{B}_E^T . Let C_4 be the matrix obtained from C_3 after subtracting the columns corresponding to (j,k') with $k' \neq k$ from the column (j,k) . The determinant of C_4 is $a+b$, and the i th entry of the (j,k) th column of C_4 is 0 if $\theta_{j,k}$ appears in p_i and -1 otherwise. Let C_5 be the matrix obtained from C_4 after subtracting from the (j,k) th column the (j,k) th column of \tilde{B}_E^T . By multilinearity of the determinant and Lemma 3.10, $\det(C_5) = b$ and C_5 is precisely the matrix obtained from \tilde{B}_E^T after replacing the column corresponding to (j,k) with the all -1 vector. ■

Proposition 3.12. *The kernel of B_E is one-dimensional. Let v be in the kernel of B_E , and let l_v be the linear polynomial whose constant term is the last entry of v and whose coefficient of $\theta_{j,k}$ is the (j,k) th entry of v . Then l_v is l_E multiplied with a scalar.*

Proof. The vector w of signed maximal minors of B_E is in the kernel of B_E by Laplace expansion. By Lemmas 3.10 and 3.11, l_E is the linear polynomial whose constant term is the last entry of w and whose coefficient of $\theta_{j,k}$ is the (j,k) th entry of w . Since l_E is not zero, not every maximal minor of B_E vanishes. Thus, its kernel is one-dimensional, and every other element of the kernel is a scalar multiple of w . ■

3.3. Computing the algebraic boundary from the Jacobian determinant. According to Proposition 3.8, the Jacobian determinant $\det(J_E)$ is the product of (3.3) and the polynomial l_E . Since we assume that the completable region has nonempty interior, we get $l_E \neq 0$. The main result of this subsection is the following theorem.

Theorem 3.13. *Eliminating the parameter variables from the ideal $G_E + \langle l_E \rangle$, where G_E is the vanishing ideal of the graph of p , gives an ideal generated by a nonconstant irreducible polynomial f . The polynomial q that defines the algebraic boundary of the completable region is the product of f with some coordinates.*

Proof. If a point on the boundary of the completable region is the image of a point of the interior of the parameter space, the inverse function theorem implies that the Jacobian determinant vanishes at this point. Since the image of p is closed in the Euclidean topology, its boundary is contained in the image of the union of the boundary of the parameter space with the zero set of the Jacobian determinant. By the assumptions on E , the boundary of the parameter space is mapped to a subset of the union of the coordinate hyperplanes. The same holds for the components of the zero set of the Jacobian determinant corresponding to the factors of (3.3). Thus, the algebraic boundary is contained in the union of the coordinate hyperplanes and the image of the zero set of l_E (which is irreducible). It cannot be contained in the union of the coordinate hyperplanes, since the image has nonempty intersection with the positive quadrant and is bounded by the hyperplane of entries summing to one. Furthermore, since the algebraic boundary is of pure codimension one by Proposition 3.3, the claim follows. ■

Remark 3.14. One could ask for a variant of Theorem 3.13 in which nonnegativity is not imposed on the domain of p . In this case the proof of Proposition 3.3 fails since the image of

p need not be closed. One could consider the same problem under the additional assumption that the map p is proper. Then the proofs show that the algebraic boundary is still given as the zero set of a factor of the product of f with the coordinates (the possibility of the factor being 1 is not excluded).

In section 4 we find the complete semialgebraic description of tensors of format $d \times \dots \times d$ with observed diagonal entries. For general E this may be too hard, but it would be interesting to understand the behavior of the degree of the boundary hypersurface.

Problem 3.15. *Determine the degree of the irreducible polynomial f in Theorem 3.13 as a function of n, d_1, \dots, d_n , and E .*

Example 3.16. We continue Example 3.4. We have

$$G_E + \langle l_E \rangle = \langle x_{211} - l_1\theta_2\theta_3, x_{121} - \theta_1l_2\theta_3, x_{112} - \theta_1\theta_2l_3, -\theta_1 - \theta_2 - \theta_3 + 2 \rangle.$$

Eliminating θ_1, θ_2 , and θ_3 yields a prime ideal generated by

$$\begin{aligned} & x_{211}^4 x_{121}^2 - 2x_{211}^3 x_{121}^3 + x_{211}^2 x_{121}^4 - 2x_{211}^4 x_{121} x_{112} + 2x_{211}^3 x_{121}^2 x_{112} + 2x_{211}^2 x_{121}^3 x_{112} \\ & - 2x_{211} x_{121}^4 x_{112} + x_{211}^4 x_{112}^2 + 2x_{211}^3 x_{121} x_{112}^2 - 6x_{211}^2 x_{121}^2 x_{112}^2 + 2x_{211} x_{121}^3 x_{112}^2 + x_{121}^4 x_{112}^2 \\ & - 2x_{211}^3 x_{112}^3 + 2x_{211}^2 x_{121} x_{112}^3 + 2x_{211} x_{121}^2 x_{112}^3 - 2x_{121}^3 x_{112}^3 + x_{211}^2 x_{112}^4 - 2x_{211} x_{121} x_{112}^4 \\ & + x_{121}^2 x_{112}^4 - 2x_{211}^3 x_{121}^2 - 2x_{211}^2 x_{121}^3 + 8x_{211}^3 x_{121} x_{112} - 4x_{211}^2 x_{121}^2 x_{112} + 8x_{211} x_{121}^3 x_{112} \\ & - 2x_{211}^3 x_{112}^2 - 4x_{211}^2 x_{121} x_{112}^2 - 4x_{211} x_{121}^2 x_{112}^2 - 2x_{121}^3 x_{112}^2 - 2x_{211}^2 x_{112}^3 + 8x_{211} x_{121} x_{112}^3 \\ & - 2x_{121}^2 x_{112}^3 + x_{211}^2 x_{121}^2 - 10x_{211}^2 x_{121} x_{112} - 10x_{211} x_{121}^2 x_{112} + x_{211}^2 x_{112}^2 - 10x_{211} x_{121} x_{112}^2 \\ & + x_{121}^2 x_{112}^2 + 4x_{211} x_{121} x_{112}. \end{aligned}$$

The zero set of this polynomial, together with the coordinate hyperplanes, is the algebraic boundary of the set of $2 \times 2 \times 2$ partial tensors with specified entries at positions $(2, 1, 1)$, $(1, 2, 1)$, and $(1, 1, 2)$ which can be completed to a rank-one tensor inside the standard simplex; see Figure 3.

Next to elimination, Sturm sequences provide another method to retrieve information about the algebraic boundary. They work directly with the image coordinates and could yield lower complexity algorithms to produce the boundary of the completable region. The nature of the construction of Sturm’s sequence warrants hope that this would yield some control over the degree in Problem 3.15. We present an example illustrating the method.

Example 3.17. As in Examples 3.4 and 3.16, consider $2 \times 2 \times 2$ partial tensors with three observed entries $x_{112}, x_{121}, x_{211}$. As argued in Example 2.11, the entry at the position $(1, 1, 1)$ is not finitely completable. Let x be an indeterminate standing for this entry. After picking x , the remaining values of the tensor all satisfy algebraic equations in the given entries and x . The slices of the tensor T then are

$$\begin{pmatrix} x & x_{121} \\ x_{211} & \frac{x_{121}x_{211}}{x} \end{pmatrix} \begin{pmatrix} x_{112} & \frac{x_{112}x_{121}}{x} \\ \frac{x_{112}x_{211}}{x} & \frac{x_{112}x_{121}x_{211}}{x^2} \end{pmatrix}.$$

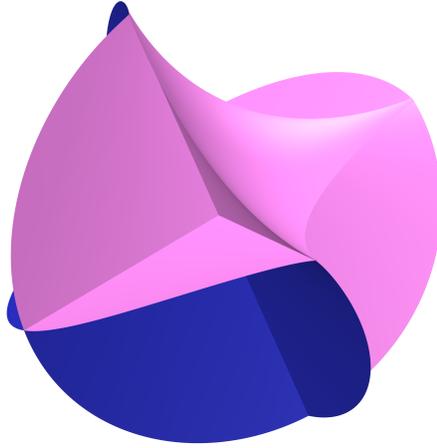


Figure 3. The irreducible surface that is part of the boundary of the completable region in Example 3.16. The completable triples inside $[0, 1]^3$ reside below the bent triangular shape. The surface is singular along the coordinate axes and the sides of the bent triangle. The algebraic boundary of the completable region also includes the coordinate hyperplanes since the bounded region below the bent triangle also extends into negative coordinates.

If T is a probability tensor, then its entries should sum to one. Let e_i be the i th elementary symmetric function on the letters $x_{112}, x_{121}, x_{211}$. This leads to the following constraint on x :

$$x + e_1 + \frac{e_2}{x} + \frac{e_3}{x^2} - 1 = 0.$$

To find conditions which guarantee the existence of real or positive solutions x we examine the Sturm sequence of this constraint after clearing denominators. The first three polynomials in the Sturm sequence are

$$\begin{aligned} f_0(x) &= \theta(x) = x^3 + (e_1 - 1)x^2 + e_2x + e_3, \\ f_1(x) &= \theta'(x) = 3x^2 + 2(e_1 - 1)x + e_2, \\ f_2(x) &= \frac{2}{9}(e_1^2 - 3e_2 - 2e_1 + 1)x + \frac{1}{9}(e_2e_1 - e_2 - 9e_3). \end{aligned}$$

The constant $f_3 = -\text{rem}(f_1, f_2)$ in the Sturm sequence is a longish quotient of two polynomials in the elementary symmetric polynomials e_1, e_2, e_3 . We omit printing it here, since it can be reproduced easily with computer algebra. To apply Sturm's theorem [31, Theorem 1.4], we evaluate at $x = 0, 1$. Assuming $e_1 \leq 1$,

$$f_0(0) = e_3 \geq 0, \quad f_1(0) = e_2 \geq 0, \quad f_2(0) = -e_2(1 - e_1) - 9e_3 \leq 0.$$

Let σ be the sign of the constant f_3 . At $x = 1$ we find

$$f_0(1) = e_1 + e_2 + e_3 \geq 0 \quad \text{and} \quad f_1(1) = 1 + 2e_1 + e_2 \geq 0.$$

Denote the sign of $f_2(1) = 2e_1^2 - 4e_1 - 7e_2 + 2 + e_1e_2 - 9e_3$ by μ . Assuming that $x_{112}, x_{121}, x_{211}$ are in the interior of Δ^2 , the sign sequence at zero is $++-\sigma$ and at one is $++\mu\sigma$.

According to Sturm’s theorem, $f_0(x)$ has a root in the half-open interval $(0, 1]$ if and only if $\mu = +$ and $\sigma = +$. Hence the completable region in the interior of Δ^2 is defined by $x_{112} > 0, x_{121} > 0, x_{211} > 0, 1 - e_1 > 0, f_2(1) \geq 0$, and $f_3 \geq 0$. By Theorem 3.13, a single irreducible polynomial in the $x_e, e \in E$ together with coordinate hyperplanes gives the algebraic boundary. Explicit computation shows that the numerator of f_3 equals a scalar multiple of the generator of the ideal in Example 3.16.

4. Completeness of diagonal partial probability tensors. We give a semialgebraic description of the region of diagonal partial tensors that can be completed to rank-one probability tensors. The following theorem is our starting point and appeared as [20, Proposition 5.2].

Theorem 4.1. *Let $E = \{(1, \dots, 1), (2, \dots, 2), \dots, (d, \dots, d)\} \subseteq [d]^n$. A diagonal partial tensor $T_E \in \mathbb{R}_{\geq 0}^E$ is completable to a rank-one tensor in Δ^{d^n-1} if and only if*

$$\sum_{i=1}^d x_{i, \dots, i}^{\frac{1}{n}} \leq 1.$$

Denote

$$S_{n,d} = \left\{ x \in \mathbb{R}_{\geq 0}^d : \sum_{i=1}^d x_{i, \dots, i}^{\frac{1}{n}} \leq 1 \right\}.$$

It was shown in [20] that $S_{n,d}$ is a semialgebraic set and a description of its algebraic boundary was given. We show for any integers $n, d \geq 1$ that the set is a basic closed semialgebraic set, and we construct the defining polynomial inequalities. We prepare some lemmata about real zeros of polynomials $f \in \mathbb{R}[t]$. To this end, let $f^{(i)}$ denote the i th derivative of f .

Lemma 4.2. *Let $f \in \mathbb{R}[t]$ be a monic polynomial of degree d . Let $\epsilon \in \mathbb{R}$ such that $f^{(i)}(\epsilon) \geq 0$ for all $i = 0, \dots, d - 1$. Then $\epsilon \geq \alpha$ for every real zero $\alpha \in \mathbb{R}$ of f .*

Proof. The statement is true when the number of real zeros (counted with multiplicity) of f is at most one because f is monic. We proceed by induction on the degree d . By the above observation the case $d \leq 1$ is clear. Let $d \geq 2$, and let f have $e \geq 2$ real zeros $\alpha_1 \leq \dots \leq \alpha_e$ (counted with multiplicity). If $\alpha_{e-1} = \alpha_e$ is a double root of f , it is also a root of f' and by the induction hypothesis, $\epsilon \geq \alpha_e$. If $\alpha_{e-1} < \alpha_e$, then by Rolle’s theorem there is a $\beta \in \mathbb{R}$ with $\alpha_{e-1} < \beta < \alpha_e$ and $f'(\beta) = 0$. Thus, by induction hypothesis, $\epsilon \geq \beta$. Since α_e is a simple root, f has a change of signs at α_e . Since f is monic, it is negative between β and α_e , and thus $\epsilon \geq \alpha_e$. ■

Lemma 4.3. *Let $f \in \mathbb{R}[t]$ be a monic polynomial of degree d . The set*

$$I = \{ \epsilon \in \mathbb{R} : f^{(i)}(\epsilon) \geq 0 \text{ for all } i = 0, \dots, d - 1 \}$$

is connected and thus a closed, unbounded interval.

Proof. Assume that there are real numbers $a < b < c$ such that $a, c \in I$ but $b \notin I$. There is a $1 \leq i < d$ such that $f^{(i)}(b) < 0$. Since $f^{(i)}(a)$ and $f^{(i)}(c)$ are nonnegative, by Rolle’s theorem and the intermediate value theorem, there is a $\xi > a$ with $f^{(i+1)}(\xi) = 0$. This contradicts $a \in I$ by Lemma 4.2 applied to $f^{(i+1)}$.

The interval I is closed because a finite number of polynomials being nonnegative is a closed condition. It is unbounded because the defining polynomials are monic and thus nonnegative for sufficiently large t . ■

Lemma 4.4. *If a polynomial f has a real zero $\alpha \in \mathbb{R}$ that is larger than the real part of any other zero of f , then*

$$\{\epsilon \in \mathbb{R} : f^{(i)}(\epsilon) \geq 0 \text{ for all } i = 0, \dots, d - 1\} = \{\epsilon \in \mathbb{R} : \epsilon \geq \alpha\}.$$

Proof. Consider the factorization of f as

$$f = \prod_{k=1}^s (t - (a_k + b_k i))(t - (a_k - b_k i)) \prod_{l=1}^r (t - c_l)$$

with $2s + r = d$ and real numbers $a_k, b_k, c_l \in \mathbb{R}$ that satisfy $a_k, c_l \leq \alpha$. Then

$$f = \prod_{k=1}^s ((t - \alpha)^2 + 2(\alpha - a_k)(t - \alpha) + (\alpha - a_k)^2 + b_k^2) \prod_{l=1}^r ((t - \alpha) + (\alpha - c_l)).$$

Thus, as a polynomial in $t - \alpha$, f has nonnegative coefficients. This shows $f^{(i)}(\alpha) \geq 0$ for all $i = 0, \dots, d - 1$, from which the statement follows by Lemmas 4.2 and 4.3. ■

Let $d, n \geq 1$ be integers. For every tuple $\sigma \in \{0, \dots, n - 1\}^d$ we define the linear polynomial $x_\sigma := \sum_{i=1}^d \zeta_n^{\sigma_i} x_i \in L[x_1, \dots, x_d]$ where $\zeta_n \in \mathbb{C}$ is a primitive n th root of unity and $L = \mathbb{Q}[\zeta_n]$. Now consider the polynomial

$$Q_{n,d} = \prod_{\sigma \in \{0, \dots, n-1\}^d} (t - x_\sigma) \in L[t, x_1, \dots, x_d].$$

Since $Q_{n,d}$ is fixed under the action of the Galois group of L over \mathbb{Q} , it has rational coefficients. Since $Q_{n,d}$ is stable under scaling t or one of the x_i by an n th root of unity, there exists a polynomial $\tilde{Q}_{n,d} \in \mathbb{Q}[t, x_1, \dots, x_d]$ of degree n^{d-1} with $Q_{n,d}(t, x_1, \dots, x_d) = \tilde{Q}_{n,d}(t^n, x_1^n, \dots, x_d^n)$. For $i = 0, \dots, n^{d-1} - 1$ let

$$P_{n,d,i} = \frac{\partial^i \tilde{Q}_{n,d}}{\partial t^i} \Big|_{t=1} \in \mathbb{Q}[x_1, \dots, x_d]$$

be the i th derivative of $\tilde{Q}_{n,d}$ evaluated at $t = 1$.

Example 4.5. Let $d = n = 2$. We have

$$\begin{aligned} Q_{2,2} &= (t - x_1 - x_2)(t - x_1 + x_2)(t + x_1 - x_2)(t + x_1 + x_2) \\ &= t^4 - 2t^2 x_1^2 - 2t^2 x_2^2 + x_1^4 - 2x_1^2 x_2^2 + x_2^4. \end{aligned}$$

As predicted, $Q_{2,2}$ is a polynomial in t^2, x_1^2, x_2^2 . We have $Q_{2,2}(t, x_1, x_2) = \tilde{Q}_{2,2}(t^2, x_1^2, x_2^2)$ with $\tilde{Q}_{2,2} = t^2 - 2tx_1 - 2tx_2 + x_1^2 - 2x_1x_2 + x_2^2$.

Theorem 4.6. *A nonnegative vector $x \in \mathbb{R}_{\geq 0}^d$ is an element of $S_{n,d}$ if and only if $P_{n,d,i}(x) \geq 0$ for all $0 \leq i < n^{d-1}$. If n is odd, then $S_{n,d} = \{x \in \mathbb{R}_{\geq 0}^d : P_{n,d,0}(x) \geq 0\}$.*

Proof. Fix $z \in \mathbb{R}_{\geq 0}^d$. The roots of $\tilde{Q}_{n,d}(t, z_1, \dots, z_d) \in \mathbb{R}[t]$ are precisely the complex numbers $(\sum_{i=1}^d \zeta_n^{\sigma_i} \sqrt[n]{z_i})^n$ for $\sigma \in \{0, \dots, n-1\}^d$. Indeed, the roots of $Q_{n,d}$ are the numbers $(\sum_{i=1}^d \zeta_n^{\sigma_i} z_i)$. Since $\tilde{Q}_{n,d}(t^n, z_1, \dots, z_d) = Q_{n,d}(t, \sqrt[n]{z_1}, \dots, \sqrt[n]{z_d})$, the zeros of $\tilde{Q}(t^n, z_1, \dots, z_d)$ are $(\sum_{i=1}^d \zeta_n^{\sigma_i} \sqrt[n]{z_i})^n$, i.e., $\tilde{Q}((\sum_{i=1}^d \zeta_n^{\sigma_i} \sqrt[n]{z_i})^n, z_1, \dots, z_d) = 0$. The real zero $\alpha = (\sum_{i=1}^d \sqrt[n]{z_i})^n \in \mathbb{R}$ is larger than the real part of any other zero. By Lemma 4.4, for every $\epsilon \in \mathbb{R}$,

$$\epsilon \geq \alpha \Leftrightarrow \frac{\partial^i \tilde{Q}_{n,d}}{\partial t^i}(\epsilon, z_1, \dots, z_d) \geq 0 \text{ for all } i = 0, \dots, n^{d-1} - 1.$$

With $\epsilon = 1$, this gives the first part of the claim. If n is odd, then α is the only real zero of $\tilde{Q}_{n,d}(t, z_1, \dots, z_d) \in \mathbb{R}[t]$, and thus $\epsilon \geq \alpha$ if and only if $\tilde{Q}_{n,d}(\epsilon, z_1, \dots, z_d) \geq 0$. ■

Let $e_{i,d}$ denote the i th elementary symmetric polynomial in x_1, \dots, x_d .

Example 4.7. Let $d = n = 2$. Then we have

$$\tilde{Q}_{2,2} = t^2 - 2t(x_1 + x_2) + (x_1 - x_2)^2.$$

Thus, $S_{2,2}$ is defined by the following inequalities:

$$\begin{aligned} x_1, x_2 &\geq 0, \\ 1 - x_1 - x_2 &\geq 0, \\ 1 - 2(x_1 + x_2) + (x_1 - x_2)^2 &\geq 0, \end{aligned}$$

which can be rewritten as

$$\begin{aligned} x_1, x_2 &\geq 0, \\ 1 - e_{1,2} &\geq 0, \\ (1 - e_{1,2})^2 - 4e_{2,2} &\geq 0. \end{aligned}$$

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