The generic initial ideal of a matroid

joint work with Matteo Varbaro and Alex Constantinescu

arXiv: *Generic and special constructions of pure O-sequences*

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Theorem

If Δ is the $(d-1)$-dimensional skeleton of d-dimensional Cohen-Macaulay complex, then the generic initial ideal $\mathrm{gin}(I_\Delta)$ is level. In particular, the h-vector of Δ is a pure O-sequence.

Definitions (Simplicial complexes)

Let Δ be a simplicial complex on $[n] := \{1, \dots, n\}$.

Stanley-Reisner ideal: $$I_\Delta = \left\langle \prod_{i \in G} x_i : G \notin \Delta \right\rangle.$$

Stanley-Reisner ring: $$\Bbbk[\Delta] := \Bbbk[x_1, \dots, x_n]/I_\Delta$$

Simplicial complexes

Δ is

- pure if all inclusion-maximal faces (:=facets) have the same size
- Cohen-Macaulay if $\mathbb{k}[\Delta]$ is a Cohen-Macaulay ring.

Reisner's criterion: Δ is Cohen-Macaulay if and only if all links in Δ have only top-dimensional homology.

Field assumptions

- Main theorem does not depend on the field.
- The Cohen-Macaulay property does depend on $\mathrm{char}(\mathbb{k})$
- in proofs we can assume whatever is convenient about $\mathbb{k}$.

Dimension convention

The Krull dimension of $\Bbbk[\Delta]$ equals the maximal size of facets of Δ. Throughout:

$$d := \dim \Bbbk[\Delta] = \dim(\Delta) + 1$$

- d is the rank of Δ.

Skeleton (Truncation)

The $(d-1)$-dimensional skeleton of Δ is $\{F \in \Delta : |F| \leq d\}$.

Why do we care about $\mathbb{k}[\Delta]$?

$\mathbb{k}[\Delta]$ encodes the combinatorics of Δ.

The finely graded Hilbert Series: sum of all monomials in $\mathbb{k}[\Delta]$

$$HS(\mathbb{k}[\Delta], (x_1, \ldots, x_n)) = \sum \{x^u : x^u \notin I_\Delta\}$$

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$$HS(\mathbb{k}[\Delta], (x_1, \ldots, x_n)) = \frac{\displaystyle\sum_{F \in \Delta} \prod_{i \in F} x_i \prod_{j \notin F} (1 - x_j)}{\prod_{i=1}^{n} (1 - x_i)}$$

The finely graded Hilbert Series: sum of all monomials in $\mathbb{k}[\Delta]$

$$HS(\mathbb{k}[\Delta], (t, \ldots, t)) = \frac{\sum_{F \in \Delta} t^{|F|}(1-t)^{n-|F|}}{(1-t)^n}$$

The finely graded Hilbert Series: sum of all monomials in $\mathbb{k}[\Delta]$

$$HS(\mathbb{k}[\Delta], t) = \frac{\sum_{k=0}^{d} f_k t^k (1-t)^{n-k}}{(1-t)^n}$$

- where $f_k = \#\{\text{faces of } \textit{size } k\}$
- $(f_0, f_1, \ldots, f_d)$ is the f-vector.

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If you compute the Hilbert series in Macaulay2 you get

$$HS(\mathbb{k}[\Delta], t) = \frac{h_d t^d + \cdots + h_1 t + h_0}{(1-t)^d}$$

$(h_0, h_1, \ldots, h_d)$ is the h-vector of Δ (trailing zeros possible).

The finely graded Hilbert Series: sum of all monomials in $\mathbb{k}[\Delta]$

$$\sum_{k=0}^{d} f_k t^k (1-t)^{d-k} = h_d t^d + \cdots + h_1 t + h_0$$

Plug in t^{-1} and get:

$$\sum_{i=0}^{d} h_i t^{d-i} = \sum_{i=0}^{d} f_i (t-1)^{d-i}$$

$\Rightarrow$ h-vector encodes face numbers!

Studying h- or f-vectors of simplicial complexes

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- Pure simplicial complexes? $\rightarrow$ Ziegler Exercise 8.16
  Don't try: It would solve all basic problems in design theory.
- Billera/Lee/Stanley: The g-theorem characterizes h-vectors of boundaries of simplicial polytopes.
  - Includes upper- and lower bound theorem.
  - open for simplicial spheres

$\Rightarrow$ Study subclasses of pure complexes.

Cohen-Macaulay complexes are pure.

Theorem (Macaulay/Stanley)

The set of h-vectors of Cohen-Macaulay complexes coincides with Hilbert functions of monomial Artinian $\mathbb{k}$-algebras (O-sequences).

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Proof

- Choose a regular sequence of linear elements l_i.
- Pass to $\mathbb{k}[\Delta]/(l_i)$ (Artinian reduction)
- Take initial ideal (to make it monomial)

What about subclasses ?

Definition

Δ is (a) matroid if for any faces $F, G \in \Delta$ with $|F| > |G|$, there exists $i \in F \setminus G$ such that $G \cup i \in \Delta$.

Matroid terminology

- $|F|$ is the rank of F.
- Facets of Δ are called bases.
- Minimal non-faces of Δ are called circuits.

Matroid are everywhere!

- $\Delta =$ sets of independent columns of a matrix.
- $\Delta =$ sets of edges of a graph not containing a circuit
- $\Delta =$ collection of sets for which the greedy algorithm finds a maximum weight set (independent of weights).
- $\ldots$

What properties should h-vectors of matroids have

Let Δ be matroid.

- $\mathbb{k}[\Delta]$ is CM $\Rightarrow h$ is an O-sequence
- $\mathbb{k}[\Delta]$ is level:

$$0 \leftarrow \mathbb{k}[\Delta] \leftarrow F_0 \leftarrow F_1 \leftarrow \ldots \leftarrow \mathbb{k}[\mathbf{x}](-a)^{\beta_p} \leftarrow 0$$

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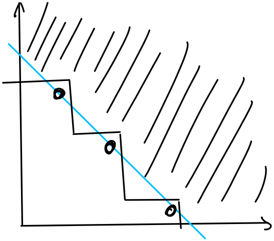

Artinian monomial algebra: all inner corners are in same degree.

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Conjecture (Stanley)

The h-vector of a matroid complex Δ is the Hilbert function of an Artinian monomial level algebra (a pure O-sequence).

Pure O-sequences

- Satisfy the Hibi inequalities:

$$h_0 \leq h_1 \leq \cdots \leq h_{\lfloor \frac{d}{2} \rfloor} \qquad h_i \leq h_{d-i} \quad 0 \leq i \leq \lfloor \frac{d}{2} \rfloor$$

  (matroid h-vectors do too, but they satisfy more)
- Need not be unimodal (some matroid h-vectors are (Huh))
- Can probably not be characterized well. See *On the shape of pure O-sequences* (BMMNZ)

All proofs so far produce the pure O-sequence explicitly.

Artinian reduction revisited

Δ matroid		Artinian	monomial	level
Stanley-Reisner ring	$\Bbbk[\Delta]$	✗	✓	✓

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Art. red. of $\mathrm{gin}(I)$	$\mathbb{k}[\mathbf{x}]/(\mathrm{gin}(I_\Delta) + x_i)$	✓	✓	?

The generic initial ideal

Fix a term order $<$

- Do random linear coordinate change on I.
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- With probability one you get the generic initial ideal $\operatorname{gin}_<(I)$.

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Properties

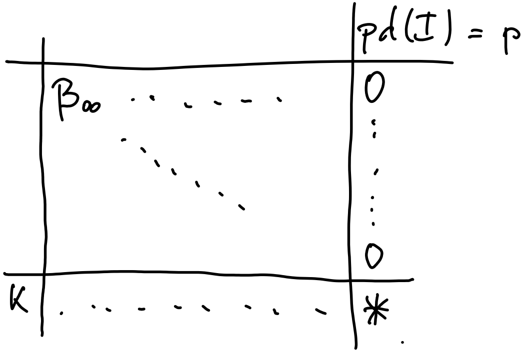
- h-vector is preserved.
- extremal Betti numbers are preserved.
- If char$(\Bbbk) = 0$, then $\mathrm{gin}(I)$ is strongly stable:
  - If $i < j$ and $x_j | m$ for some $m \in \mathrm{gin}(I)$, then $\frac{x_i}{x_j} m \in \mathrm{gin}(I)$.
- Eliahou-Kervaire resolution gives formulas for the Betti numbers.
- admits regular sequence of variables.

When is $\mathrm{gin}(I_\Delta)$ level?

Lemma 1

Let $I \subseteq \Bbbk[\mathbf{x}]$ be graded and $\operatorname{reg}(I) = k$. If $\operatorname{pd}(I_{<k}) < \operatorname{pd}(I) =: p$

$$\beta_p(I) = \beta_{p,p+k}(I) = \beta_{p,p+k}(\operatorname{gin}(I)) = \beta_p(\operatorname{gin}(I))$$



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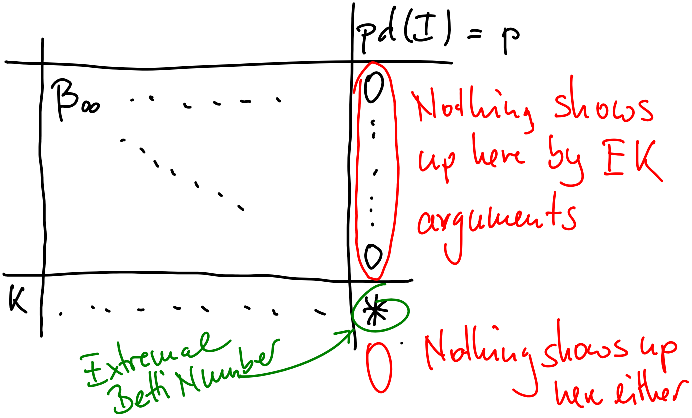

Lemma 2

If Δ is CM of rank $d+1$ and F a minimal non-face of size $d+1$, then $\Delta \cup F$ is CM.

$$0 \to \Bbbk[\Delta \cup F] \to \Bbbk[\Delta] \oplus \Bbbk[F] \to \Bbbk[\Delta \cap F] \to 0$$

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Theorem

If Δ is the rank d skeleton of some rank $d+1$ Cohen-Macaulay complex, then the generic initial ideal $\mathrm{gin}(I_\Delta)$ is level.

Proof

- $\mathrm{reg}(I) - 1 = \mathrm{reg}(\Bbbk[\Delta]) = d$
  (Hochster: $\leq d$, generator in degree $d+1$)

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  (Hochster: $\leq d$, generator in degree $d+1$)
- Let $J = (I_\Delta)_{\leq d}$ and $\Gamma = \Delta(J)$.
  - For Lemma 1 need $\mathrm{pd}(\Bbbk[\Gamma]) < \mathrm{pd}(\Bbbk[\Delta])$
  - $\Leftrightarrow$ depth$(\Bbbk[\Gamma]) > d+1$
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  - $\Leftrightarrow \Gamma_d :=$ rank $(d+1)$ skeleton of Γ is CM
- Let Ω be the CM complex of which Δ is the skeleton.
- Facets of Γ_d are each either facets of Ω or minimal non-faces of size $d+1$ (by construction of Γ).
- Lemma 2 $\Rightarrow \Gamma_d$ is CM. $\square$

How many matroids are truncations of other matroids?

Example

No complete bipartite graph (matroid of rank 2) is a truncation.

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Example

If a rank d matroid Δ is a truncation, then it is a truncation of some rank $d+1$ matroid Γ.

- Any facet of Γ is a *spanning circuit* ($:=$ size $d+1$) of Δ.

There are (many) matroids without spanning circuits!

Brylawski's algorithm

Let Δ be a matroid with a spanning circuit. Brylawski's algorithm decides if Δ is a truncation of some Γ and constructs the *freest* possible Γ.

Example

Schubert matroids (generalized Catalan matroids $=$ PI matroids) have componentwise linear I_Δ and in particular satisfy Stanley's conjecture.

Δ matroid		Artinian	monomial	level
Stanley-Reisner ring	$\Bbbk[\Delta]$	✗	✓	✓
Artinian reduction	$\Bbbk[\Delta]/(l_i)$	✓	✗	✓
Art. red. of $\mathrm{gin}(I)$	$\Bbbk[\mathbf{x}]/(\mathrm{gin}(I_\Delta) + x_i)$	✓	✓	?

Wish

		Artinian	monomial	level
weak gin	$\Bbbk[\mathbf{x}]/(\text{weakgin}(I_\Delta) + (x_i - x_j)$	✓	✓	✓

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A large example

$$h = (1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 112, 116, 111, 96, 70, 40, 14)$$

is the h-vector of a matroid on 20 vertices.

Our method proves: h equals the Hilbert function of the Artinian monomial level algebra $\mathbb{k}[a, b, c, d]/I$ where

$$I = \big(a^{10}, a^6b^4, a^3b^{10}, ab^{13}, b^{15}, a^3b^4c^3, b^{11}c^3, a^6c^5, ab^4c^5, b^5c^5, ac^9, b^2c^{10}, \\ c^{16}, ad, b^9d, b^5c^4d, c^{13}d, b^2c^4d^4, c^{11}d^4, b^5d^6, c^7d^6, b^2d^{10}, c^3d^{10}, d^{14}\big).$$

Parallel elements

Two elements i, j are parallel in Δ if $\{i, j\}$ is a circuit.

Dual matroid

The matroid dual of Δ has the complements of facets of Δ as facets.

Theorem

- Stanley's conjecture holds for matroids of CM type at most five.
- Stanley's conjecture holds if dual has (rank+2) parallel classes.

The search for a counterexample

- Matroids on nine or fewer vertices satisfy Stanley's conjecture (deLoera/Kemper/Klee)
- Type must be at least 6.
- To confirm a counterexample, need to check $\binom{N}{6}$ possible socles where $N = \binom{n+s-1}{n-1}$ is a binomial coefficient.
- Hard to exploit symmetry (checking a single socle is quick).

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