How primary decomposition of binomial ideals and monoid congruences is wrong

Thomas Kahle (ETH Zürich)
(joint with Ezra Miller, Duke)

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No Theorem

Let $\mathbb{k}$ be any field and Q a commutative Noetherian monoid. Every binomial ideal in $\mathbb{k}[Q]$ has a decomposition into primary binomial ideals.

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Theorem (Lasker/Noether 1905/1921)

Let $\mathbb{k}$ be any field and Q a commutative Noetherian monoid. Every binomial ideal in $\mathbb{k}[Q]$ has a decomposition into primary ideals.

Theorem (Eisenbud/Sturmfels 1996)

Let $\mathbb{k}$ be any algebraically closed field and Q a commutative Noetherian monoid. Every binomial ideal in $\mathbb{k}[Q]$ has a decomposition into primary binomial ideals.

"If one has never done any calculations, one would be inclined to say – let's extend $\mathbb{k}$ as far as needed to split our algebraic set. *That is a very bad idea!*"

Dave Bayer / David Mumford
in "What can be computed in algebraic geometry"

Monoids

- In this talk: $(Q, +)$ is a commutative Noetherian monoid.

Monoid Algebra

Let $\mathbb{k}$ be a field. The monoid algebra over Q is the $\mathbb{k}$-vector space

$$\mathbb{k}[Q] := \bigoplus_{q \in Q} \mathbb{k}\left\{\mathbf{t}^q\right\} \qquad \text{with} \qquad \mathbf{t}^q\mathbf{t}^u := \mathbf{t}^{q+u}.$$

A binomial ideal is an ideal generated by binomials

$$\mathbf{t}^q - \lambda\mathbf{t}^u, \quad q, u \in Q, \lambda \in \mathbb{k}.$$

Polynomial rings

$\mathbb{k}[x_1, \ldots, x_n] = \mathbb{k}[\mathbb{N}^n]$

$\rightarrow$ Fix the no theorem combinatorially.

Congruence basics

Definition

A congruence on Q is an equivalence relation $\sim$ such that

$$a \sim b \Rightarrow a + q \sim b + q \quad \forall q \in Q$$

The quotient $\overline{Q} := Q/\sim$ is a monoid again.

Congruences from binomial ideals

Each binomial ideal $I \subseteq \Bbbk[Q]$ induces a congruence $\sim_I$ on Q:

$$a \sim_I b \Leftrightarrow \exists \lambda \neq 0 : \mathbf{t}^a - \lambda \mathbf{t}^b \in I$$

Monomial ideals?

Let $T \subseteq Q$ be a monoid ideal. The monomial ideal $\langle \mathbf{t}^q : q \in T \rangle$ induces the Rees congruence identifying all elements of T.

Congruence basics

Special congruences

- The identity congruence: $\{(q,q) : q \in Q\}$: $\overline{Q} = Q$
- The universal congruences: $Q \times Q$: $\overline{Q} = \{0\}$.

Lemma

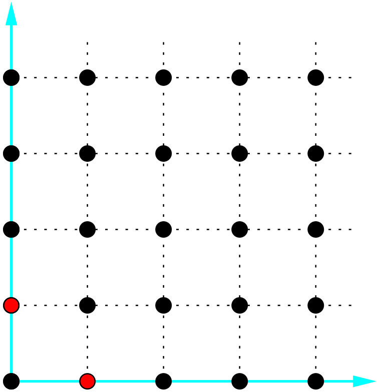
- Congruences are exactly the kernels of monoid morphisms:

$$\ker \phi = \big\{(u,v) \in Q^2 : \phi(u) = \phi(v)\big\}$$

- Every commutative Noetherian monoid is presented as $Q = \mathbb{N}^n/\sim$.

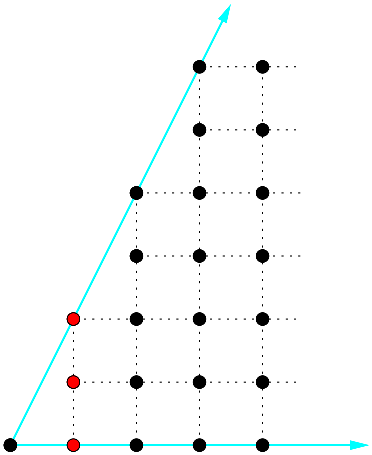
$\rightarrow$ understand monoids really well

The mother of all monoids: $\mathbb{N}^n$

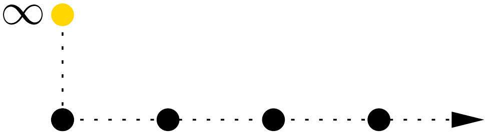


$\mathbb{N}^2$

Affine semigroups



Weird monoids

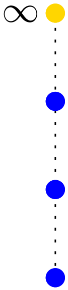


$\mathbb{N} \cup \{\infty\}$

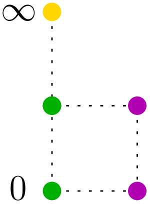
Nil

An element $q \in Q$ is nil if $q + a = q$ for all $a \in Q$.

Weird monoids

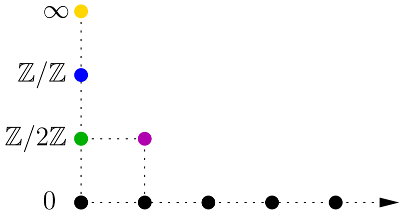


$\mathbb{N}/(3+\mathbb{N})$

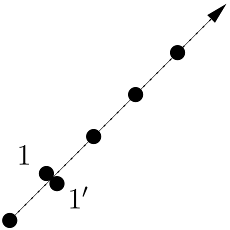


$(\mathbb{Z}/2\mathbb{Z} \times \mathbb{N})/(2+\mathbb{N})$

Weird monoids

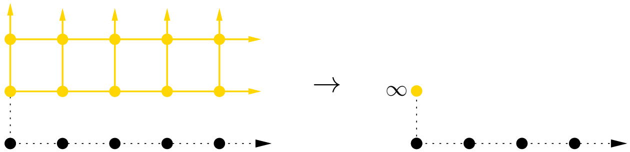


A weird monoid

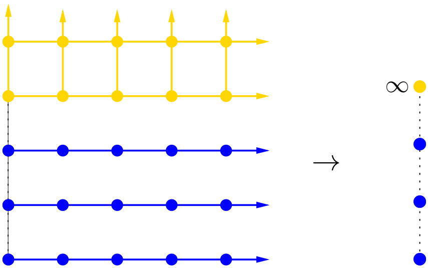


$\mathbb{N}$ with one doubled

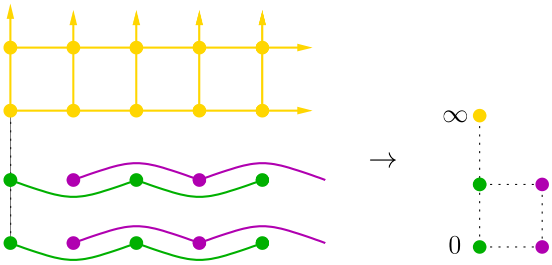
Presenting weird monoids



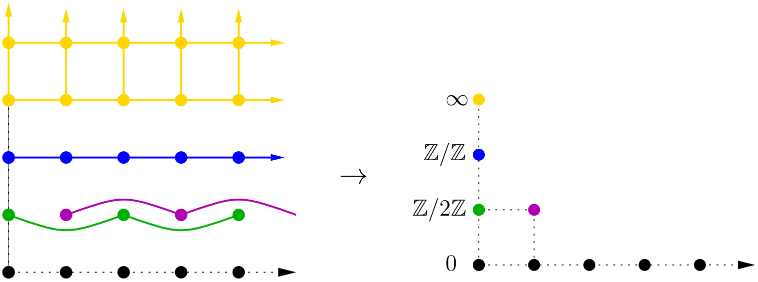
Presenting weird monoids



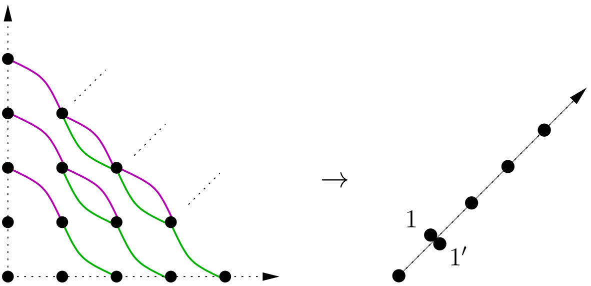
Presenting weird monoids



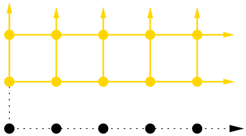
Presenting weird monoids



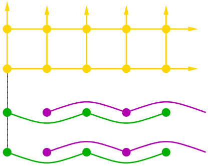
Presenting weird monoids



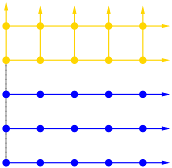
Some binomial ideals



$\langle y \rangle$

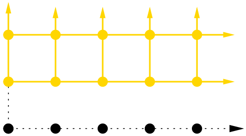


$\langle y^2, x^2 - 1 \rangle$



$\langle y^3, x - 1 \rangle$

Some binomial ideals



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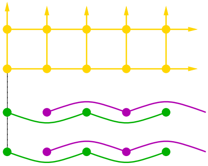

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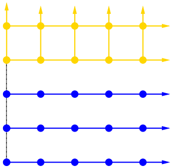
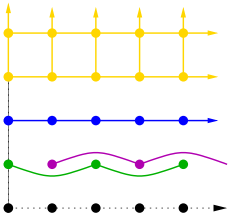
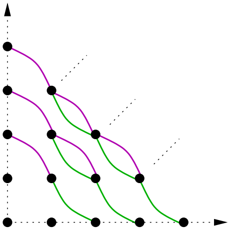


$\langle y^3, x - 1 \rangle$



$\langle y^3, y^2(x-1), y(x^2-1) \rangle$



$\langle x^2 - xy, xy - y^2 \rangle$

Monoid elements

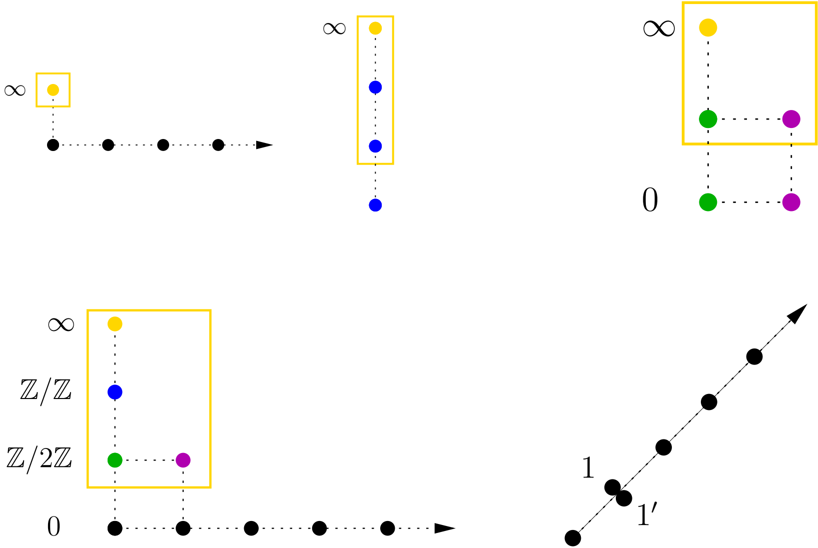
Monoid elements

An element $q \in Q$ is

- cancellative if $a + q = b + q \Rightarrow a = b$, for all $a, b \in Q$.
- nilpotent if a multiple is nil.

monomial $\mathbf{t}^q \in \Bbbk[Q]$	monoid element $q \in Q$
nonzerodivisor	cancellative
"nilpotent"	nilpotent

Nilpotents



Formal analogy

Definition

An ideal $I \subseteq \Bbbk[Q]$ is

- prime if in $\Bbbk[Q]/I$ every element is either zero or a nonzerodivisor.
- primary if in $\Bbbk[Q]/I$ every element is either nilpotent or a nonzerodivisor.
- irreducible if it is not the intersection of two strictly larger ideals.

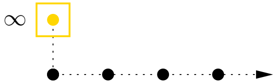
Definition (Drbohlav, 1963)

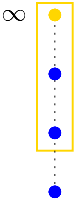
A congruence $\sim$ on Q is

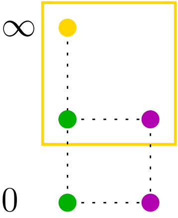
- prime if in $Q/\sim$ every element is either nil or cancellative.
- primary if in $Q/\sim$ every element is either nilpotent or cancellative.
- irreducible if it is not the common refinement of two strictly coarser congruences

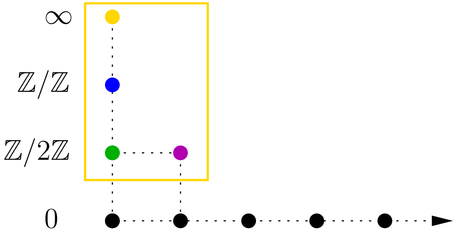
$\Rightarrow$ Congruences have primary decompositions.

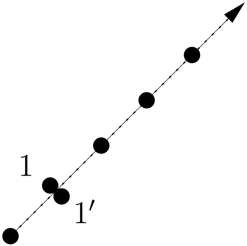
Primary quotients



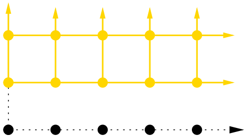
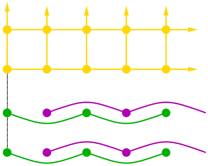
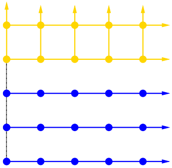




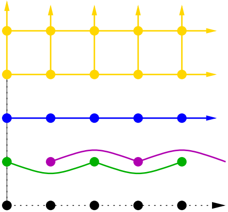




Primary congruences

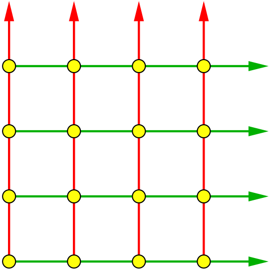
Wants to be decomposed



Conclusion

- Primary decomposition of congruences is too coarse.

Prime congruences may be reducible



The identity congruence on $\mathbb{N}^2$ has a primary decomposition

Conclusion

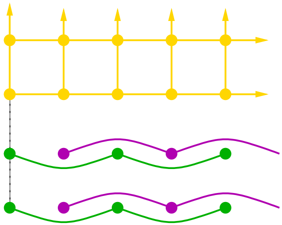
- Primary decomposition of congruences is too fine

Mesoprimary congruences

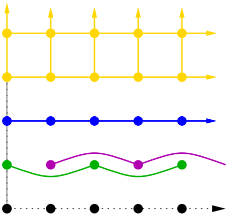
Definition

- An element $q \in Q$ is partly cancellative if $a + q = b + q$ implies $a = b$ or $a + q = \infty$, whenever a and b differ by a cancellative.
- A congruence is mesoprimary if it is primary and in $Q/\sim$ every element is partly cancellative.

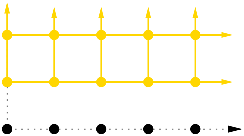
mesoprimary



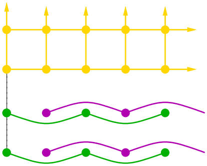
not mesoprimary



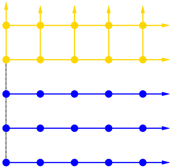
Mesoprimary decomposition



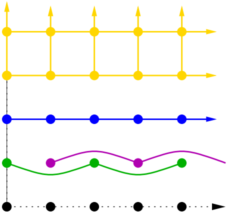
$\langle y \rangle$



$\left\langle x^2 - 1, y^2 \right\rangle$



$\left\langle x - 1, y^3 \right\rangle$



$\left\langle y(x^2 - 1), y^2(x - 1), y^3 \right\rangle$

How to find components?

Localizations of monoids at prime ideals

Prime ideals in monoids

- Q has only finitely many prime ideals.
- $\varnothing \subseteq Q$ is an ideal (think: $\langle 0 \rangle \subseteq \Bbbk[Q]$)

Localization

Let $P \subseteq Q$ be a prime ideal, and $\sim$ a congruence on Q.

- The localization Q_P of Q at P is the monoid arising from adjoining inverses for all elements not in P.
- The induced congruence on Q is also denoted $\sim$, and $\overline{Q}_P := Q_P/\sim$.

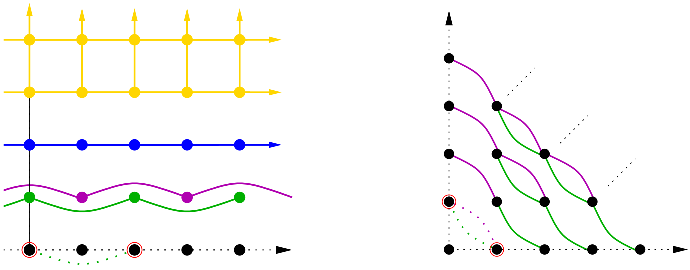
Example

- Localizing a monoid Q without nil at $\varnothing$ gives its universal group $Q_\varnothing$.

Associated prime congruences

Detecting combinatorial changes

1. Localize at a prime $P \subseteq Q$
2. Detect witnesses: socle elements in Q_P
   - check kernels of addition morphisms $\phi_p : q \mapsto q + p,\ p \in P_P$



A prime congruence is associated if its 1 “looks like” a witness.

Coprincipal and mesoprimary components

Witnessed decompositions

- The congruence $\sim$ defines a set of witnesses (P, w)
- For each witness (P, w):
  1. Localize at P.
  2. Lemma: The nilpotent quotient $\overline{Q}_P$ is partially ordered.
  3. Make every class below w look like w.
  4. Join to ∞ every class not below w.

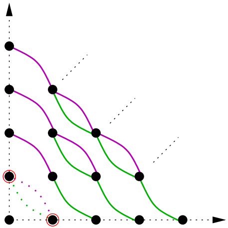
Result: coprincipal component, a mesoprimary congruence that looks like $\sim$ at w and imposes no further conditions.

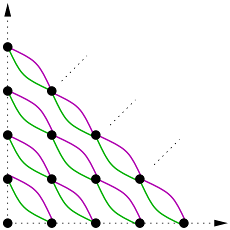
Coprincipal mesoprimary decomposition

- Decompose using coprincipal components
- Components at key witnesses suffice

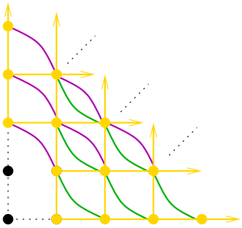
Coprincipal mesoprimary decomposition

$$\langle x^2 - xy, xy - y^2 \rangle$$

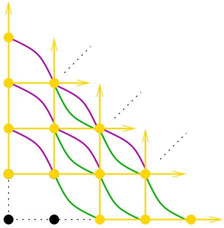




$\langle x - y \rangle$



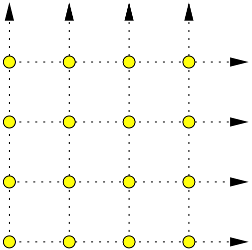
$\langle x^2, y \rangle$



$\langle x, y^2 \rangle$

No coprincipal decomposition

Nothing to decompose:



Mesoprimary decomposition of congruences

Theorem

Every congruence $\sim$ on Q is the common refinement of mesoprimary congruences that are coprincipal components at key witnesses.

Mesoprimary decomposition of congruences

- is canonical
- need not be irredundant
- fixes deficiencies of irreducible decomposition

Lifting decompositions to $\mathbb{k}[Q]$

Mesoprimary decompositions of binomial ideals

- Determine witnesses of $\sim_I$.
- Coprincipal components are inducing coprincipal components of $\sim_I$.
- Decompose using coprincipal components for character witnesses

Subtleties in the lifting procedure

- $\langle x - 1\rangle \cap \langle y - 1\rangle$ is not binomial.
  - There are character witnesses that are not key.
- $\langle z - 1\rangle \cap \langle z + 1\rangle \neq \langle z - 1\rangle \cap \langle z - 1\rangle$
  - There are false witnesses among the key witnesses.

- Mesoprimary decomposition
  - of congruences uses key witnesses
  - of binomial ideals uses character witnesses

Mesoprimary decomposition of binomial ideals

Theorem

Let $\mathbb{k}$ be any field. Every binomial ideal $I \subseteq \mathbb{k}[Q]$ has a mesoprimary decomposition into binomial ideals that are coprincipal components of I at character witnesses.

Mesoprimary decomposition of binomial ideals

- is canonical
- need not be irredundant
- yields irreducible decomposition of binomial ideals

Even if $\mathbb{k} = \mathbb{C}$

Do you really...

want your computer to decompose $\langle x^{17} - 1, y^2 \rangle$?

```
{ideal(x-1,y^2), ideal(x-ww_17,y^2), ideal(x-ww_17^2,y^2), ideal(x-ww_17^3,y^2),
ideal(x-ww_17^4,y^2), ideal(x-ww_17^5,y^2),ideal(x-ww_17^6,y^2), ideal(x-ww_17^7,y^2),
ideal(x-ww_17^8,y^2), ideal(x-ww_17^9,y^2), ideal(x-ww_17^10,y^2), ideal(x-ww_17^11,y^2),
ideal(x-ww_17^12,y^2), ideal(x-ww_17^13,y^2), ideal(x-ww_17^14,y^2), ideal(x-ww_17^15,y^2)
ideal(x+ww_17^15+ww_17^14+ww_17^13+ww_17^12+ww_17^11+ww_17^10+ww_17^9+ww_17^8+
                 ww_17^7+ww_17^6+ww_17^5+ww_17^4+ww_17^3+ww_17^2+ ww_17+1,y^2)}
```

Even if $\mathbb{k} = \mathbb{C}$

Do you really...

want your computer to decompose $\langle x^{17} - 1, y^2 \rangle$?

```
{ideal(x-1,y^2), ideal(x-ww_17,y^2), ideal(x-ww_17^2,y^2), ideal(x-ww_17^3,y^2),
ideal(x-ww_17^4,y^2), ideal(x-ww_17^5,y^2),ideal(x-ww_17^6,y^2), ideal(x-ww_17^7,y^2),
ideal(x-ww_17^8,y^2), ideal(x-ww_17^9,y^2), ideal(x-ww_17^10,y^2), ideal(x-ww_17^11,y^2),
ideal(x-ww_17^12,y^2), ideal(x-ww_17^13,y^2), ideal(x-ww_17^14,y^2), ideal(x-ww_17^15,y^2)
ideal(x+ww_17^15+ww_17^14+ww_17^13+ww_17^12+ww_17^11+ww_17^10+ww_17^9+ww_17^8+
                  ww_17^7+ww_17^6+ww_17^5+ww_17^4+ww_17^3+ww_17^2+ ww_17+1,y^2)}
```

Thank you for your attention.