Markov bases of narrow box piles

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Motivation:

Markov bases are often huge but with few combinatorial patterns.

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This morning

$3 \times 3 \times k$ tables: Markov basis stabilizes at $k = 5$.

Some families of Markov bases stabilize as parameters grow.

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No hope theorem (De Loera/Onn)

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... but there is hope

The independent set theorem (Hillar/Sullivant)

If in a graph model the variable cardinalities diverge for an independent set, then there is stabilization of Markov bases.

Infinite two-way tables

Consider $(k \times k)$-matrices with zeros along the diagonal.

- Let ϕ be the map that computes row sums and column sums.
- Let ψ be the map that adds a times the row sums to b times the column sums.
- Let $\pi = \psi \circ \phi$ be the composition.

Wanted: Limit Markov basis of $\ker(\pi)$, modulo symmetry $(k \to \infty)$.

Commutative algebra

Let

- $K[X] := K[x_i : i \in \mathbb{N}]$,
- $K[Y] := K[y_{ij} : i \neq j \in \mathbb{N}]$,
- $K[Z] := K[z_{1i}, z_{2i} : i \in \mathbb{N}]$.

Consider the monomial map

$$\pi : K[Y] \to K[X] \qquad y_{ij} \mapsto x_i^a x_j^b.$$

This map splits:

$$\pi : K[Y] \xrightarrow{\phi} K[Z] \xrightarrow{\psi} K[X],$$

with $\phi(y_{ij}) = z_{1i}z_{2j}$ and $\psi(z_{1i}) = x_i^a, \psi(z_{2i}) = x_i^b$.

For the algebraic statisticians

The image of ϕ is the independence model with structural zeros along the diagonal. Its Markov basis is known and consists of quadrics from the independence model

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \qquad y_{12}y_{34} - y_{14}y_{23},$$

and cubics

$$\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \qquad y_{12}y_{23}y_{31} - y_{13}y_{21}y_{32}.$$

Symmetry

The group $S_\infty := \bigcup_j S_j$ acts on indices:

$$\sigma(y_{ij}) = y_{\sigma(i)\sigma(j)} \quad \sigma(x_i) = x_{\sigma(i)} \quad \sigma(z_{si}) = z_{s\sigma(i)}.$$

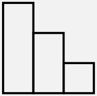
- Aschenbrenner/Hillar: $K[X]$ is equivariantly Noetherian: Every ideal closed under the action of S_∞ is generated by finitely many orbits. Note: $K[Y]$ is not.
- Draisma et al.: For many monomial maps into equi-Noetherian rings the kernel is finitely generated up to symmetry.

Represent monomials in the various rings as *box piles*.

Box Piles

Let's represent a power of a variable $x_i^{d_i}$ by a box of height d_i in column i of some display. E.g.

$$x_1^3 x_2^2 x_3 \quad \leftrightarrow$$



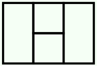
- Box piles represent monomials in $K[X]$.
- S_∞ acts $\Rightarrow$ We don't need to label columns.

Monomials in $K[Z]$

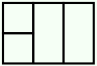
Fix integers $a > b$.

- $z_{1i} \Rightarrow$ box of height a in column i.
- $z_{2i} \Rightarrow$ box of height b in column i.

Example: $a = 2, b = 1$



$z_{11}z_{13}z_{22}^2$



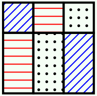
$z_{12}z_{13}z_{21}^2$

Since both piles have the same shape,

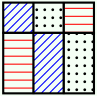
$$\psi(z_{11}z_{13}z_{22}^2 - z_{12}z_{13}z_{21}^2) = 0$$

Monomials in $K[Y]$

- Represent a variable as two colored boxes: $y_{12} \leftrightarrow$ .
- A monomial is a colorful box pile.
- The specific pattern does not matter, just the matching (never match in the same column).



$y_{12}y_{23}y_{31}$



$y_{21}y_{32}y_{13}$

This shows that

$$\pi(y_{12}y_{23}y_{31} - y_{21}y_{32}y_{13}) = 0$$

The Markov basis problem in this formalism

For each two colorful box piles with the same outer shape, there exists a sequence of Markov moves that turns one into the other.

Theorem (K/Krone/Leykin)

Let $a > b$ be coprime. The following is a Markov basis of $\ker(\pi)$.

(i) $y_{12}y_{34} - y_{14}y_{32}$;

(ii) $y_{12}y_{23}y_{31} - y_{21}y_{32}y_{13}$;

(iii) for each $0 \leq n \leq a-b$,

$$y_{12}^{b+n} \prod_{j \geq 3} y_{j2}^{c_{1j}} y_{2j}^{c_{2j}} - y_{21}^{b+n} \prod_{j \geq 3} y_{j1}^{c_{1j}} y_{1j}^{c_{2j}}$$

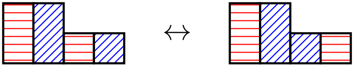
where $\sum_{j \geq 3} c_{1j} = a - b - n$ and $\sum_{j \geq 3}^{\infty} c_{2j} = n$;

(iv) for each $1 \leq n \leq b$,

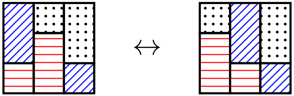
$$y_{12}^{b-n} y_{13}^{n} y_{32}^{a-b+n} - y_{21}^{b-n} y_{23}^{n} y_{31}^{a-b+n}.$$

- degree: $\max(a+b, 2a-b)$.
- width: $\max(4, a-b+2)$.

Type (i): Basic quadric



Type (ii): Three-cycle cubic



Type (iii): For each $0 \leq n \leq a - b$,

$$y_{12}^{b+n} \prod_{j \geq 3} y_{j2}^{c_{1j}} y_{2j}^{c_{2j}} - y_{21}^{b+n} \prod_{j \geq 3} y_{j1}^{c_{1j}} y_{1j}^{c_{2j}}$$

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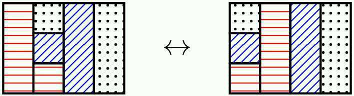
where $\sum_{j \ge 3} c_{1j} = a-b-n$ and $\sum_{j \ge 3}^{\infty} c_{2j} = n$:

$a = 3, b = 1, n = 0, c_{13} = c_{14} = 1$



$y_{12} y_{32} y_{42} - y_{21} y_{31} y_{41}$

Type (iii): For each $0 \leq n \leq a - b$,

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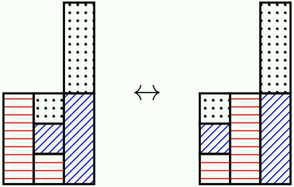
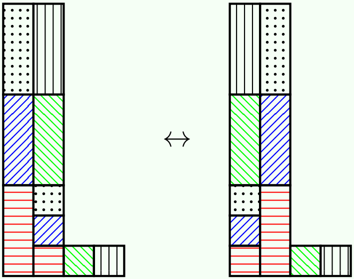
where $\sum_{j \geq 3} c_{1j} = a - b - n$ and $\sum_{j \geq 3}^{\infty} c_{2j} = n$:

$a = 3, b = 1, n = 0, c_{13} = 2$



$$y_{12} y_{32}^2 - y_{21} y_{31}^2$$

$a = 3, b = 1, n = 2, c_{23} = c_{24} = 1$

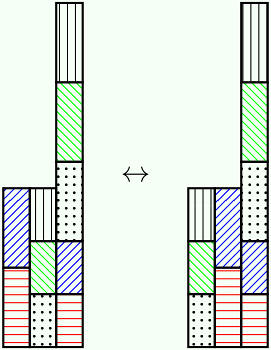


$$y_{12}^3 y_{23} y_{24} - y_{21}^3 y_{13} y_{14}$$

Type (iv):

For each $1 \leq n \leq b$:
$$y_{12}^{b-n} y_{13}^{n} y_{32}^{a-b+n} - y_{21}^{b-n} y_{23}^{n} y_{31}^{a-b+n}.$$

$a = 3, b = 2, n = 2$



$$y_{13}^{2} y_{32}^{3} - y_{23}^{2} y_{31}^{3}$$

Extension to width k?

Width k:

$$y_{i_1 \dots i_k} \mapsto x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k}$$

- Equivariant Markov basis seems complicated (though it exists).
- We give an equivariant *lattice generating sets*:
  - width at most $k+2$ with $(k^2+k-2)/2$ elements, or
  - width $2k$ with $k-1$ elements.
- Improves upon Hillar/del Campo's width $2(\sum_i a_i) - 1$.

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Thank you for your attention.

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