Boij-Söderberg theory: Cones of homological invariants

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Graded modules and Betti numbers

\[ S = k[x_1, \ldots, x_n]. \]
Always finitely generated graded modules.

\[ M \text{ a graded module } \leadsto \text{ graded Betti numbers } \beta_{ij}(M). \]

\[ \beta = \{\beta_{ij}(M)\} \in \mathbb{Q}^{[0,n] \times \mathbb{Z}} \]

The \( \beta \) generate a positive cone \( C^{\text{betti}}(\text{mod}, n) \) in \( \mathbb{Q}^{[0,n] \times \mathbb{Z}} \).
Subcategories

\textbf{mod} is the category of \textit{f.g. graded} $S$-modules. 
\textbf{$\mathcal{M}$} is an additive subcategory of \textbf{mod}.

Get subcone $C^{betti}(\mathcal{M}, n)$ of $C^{betti}(\text{mod}, n)$.
Example subcategories

Example (Of subcategories $\mathcal{M}$)

- $\text{CM}^c$, Cohen-Macaulay (CM) modules of codimension $c$.
- $\text{modArt}_0$, artinian modules generated in degree 0.
- $\text{mod}_0, m$, modules generated in degree 0, $m$-regular.
- $\text{Sq}$, squarefree modules. (Natural module category of $\mathbb{N}^n$-graded modules containing Stanley-Reisner ideals and rings.)
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If $\mathcal{A}$ is an additive category, denote by $K\mathcal{A}$ the category of complexes of objects in $\mathcal{A}$. 
Extremal rays

The cones are described by their extremal rays.

**Theorem (Boij-Söderberg, Eisenbud-Schreyer)**

*The extremal rays in $C^{\text{betti}}(\text{mod}, n)$ are given by Betti diagrams of pure resolutions of CM modules*

$$S(-d_0)^{\beta_0,d_0} \leftarrow \cdots \leftarrow S(-d_p)^{\beta_p,d_p},$$

$p \in [0, n]$. *All such pure resolutions exist.*
Cones of Hilbert functions

Hilbert function $h_j(M) = \dim_k M_j \rightsquigarrow H = \{h_j\} \in \mathbb{Q}^\mathbb{N}$.

$C_{\text{hilb}}(\mathcal{M}, n)$ subcone of $\mathbb{Q}^\mathbb{N}$ generated by Hilbert functions of modules $M$ in $\mathcal{M}$. 
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**Theorem (M. Boij-G. Smith)**

*The extremal rays in $C^{\text{hilb}}(\text{modArt}_0, n)$ are given by Hilbert functions of $S/\langle x_1, \cdots, x_n \rangle^i, \ i \geq 1$. The extremal rays of $C^{\text{hilb}}(\text{mod}_m, n)$ are also described.*
$\mathcal{F}$ coherent sheaf of dimension $\leq d$ on a projective space $\mathbb{P}$
$\rightsquigarrow$ graded cohomology $\gamma_{ij}(\mathcal{F}) = \dim_k H^i(\mathbb{P}, \mathcal{F}(j))$
$\rightsquigarrow \gamma = \{\gamma_{ij}\} \in \mathbb{Q}[0,d] \times \mathbb{Z}$. 
Cones of cohomology tables

$\mathcal{F}$ coherent sheaf of dimension $\leq d$ on a projective space $\mathbb{P}$
$\mapsto$ graded cohomology $\gamma_{ij}(\mathcal{F}) = \dim_k H^i(\mathbb{P}, \mathcal{F}(j))$
$\mapsto \gamma = \{\gamma_{ij}\} \in \mathbb{Q}[0,d] \times \mathbb{Z}$.

Such $\gamma$ generate a subcone $C'(\text{coh}_{\mathbb{P}}, d)$ of $\mathbb{Q}[0,d] \times \mathbb{Z}$. 
Regularity

A complex of coherent sheaves $\mathcal{F}^\bullet$ on a projective space $\mathbb{P}$ is $m$-regular if every homology sheaf $H^i(\mathcal{F}^\bullet)$ is $m$-regular.
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Consider complexes of coherent sheaves $\mathcal{F}^\bullet$ such that:

- $\mathcal{F}^\bullet$ is 1-regular
- The derived dual $\mathbb{R}\text{Hom}_{\mathbb{P}}(\mathcal{F}^\bullet, \omega_{\mathbb{P}})$ is $n + 1$-regular. (This implies $\dim \text{Supp} H^i(\mathcal{F}^\bullet) \leq n + 1$.)
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$\therefore$ Cohomology $\gamma = \{ \gamma_{ij} \} \in \mathbb{Q}[0,n] \times \mathbb{Z}$ and subcones

$$C^{\text{cohom}}(\text{coh}_\mathbb{P}, n) \subseteq C^{\text{cohom}}(K\text{coh}_\mathbb{P}, n) \subseteq \mathbb{Q}[0,n] \times \mathbb{Z}.$$
Homological data

$F_\bullet$ a complex of free $S$-modules. It has three sets of homological invariants:

**Homology:**  
$h_{ij} = \dim_k H_i(F_\bullet)_j \quad H = \{h_{ij}\}$

**Betti:**  
$F_i = \bigoplus S(-j)^{\beta_{ij}} \quad B = \{\beta_{ij}\}$

**Cohomology:**  
$c_{ij} = \dim_k H_i(\text{Hom}(F_\bullet, \omega_S))_j \quad C = \{c_{ij}\}$

**Note.**  
$\omega_S = S(-n)$. 
Cones of homological data

- Triplets \((H, B, C)\) generate a positive cone in \((\mathbb{Q}^{\mathbb{Z} \times \mathbb{Z}})^3\)

\[
C^{\text{trip}}(K\text{Freemod}, n).
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- Also the analog for squarefree modules
  \[ C^{\text{trip}}(K\text{FreeSq}, n) \subseteq (\mathbb{Q}^{\mathbb{Z} \times [0,n]})^3. \]
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- If $CM_c$ is the subcone of $K\text{Freemod}$ consisting of free resolutions of $CM$ modules of codimension $c$, the projection

  \[ C^{\text{trip}}(CM^c, n) \xrightarrow{\cong} C^{\text{betti}}(CM^c, n) \]

  is an isomorphism.
Describe these cones. What are the extremal rays?
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The extremal rays of $\text{C}^{\text{hilb}}(\text{mod}_{0,m}, n)$ are also described.
Resolutions of an ideal $I$.

- If $I = I_X$, Betti numbers reflect geometric properties of variety $X$: Clifford index for curves, Greens conjecture.
- If $I = I_\Delta$, Stanley-Reisner ideal, Betti numbers reflect combinatorial/homological properties of $\Delta$.
- Similar for modules $M$ which are “close” to $I$ and of $S/I$, modules of low rank or degree.
Nature of problem II

Cones reflect what happens in the “limit” for Betti numbers, Hilbert functions, cohomology tables, as the “size” (i.e. degree/rank) of the module goes to infinity.
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An analog of this is stable homotopy theory in algebraic topology, where one considers spectra, “limits” of $S^p \wedge X$ as $p \to \infty$ and the “limit” stable homotopy groups.
It is conjectured that every variety $X$ of dimension $d$ in a projective space, has an Ulrich sheaf, i.e. a coherent sheaf with the same cohomology table as $\mathcal{O}_{\mathbb{P}^d}$, up to a scalar multiple.

If this holds, the cones $C^{\text{coh}}(\text{coh}_X, n)$ are the same for all embedded varieties $X$ in a projective spaces.
Characterize the following cones, i.e. their extremal rays:

- $C^\text{cohom}(K\text{coh}_P, n)$
Main objectives

Characterize the following cones, i.e. their extremal rays:

- \( C^{\text{cohom}}(\text{Kcoh}_F, n) \)
- \( C^{\text{trip}}(\text{KFremod}, n) \)
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- $C^{\text{trip}}(K\text{Freemod}, n)$
- $C^{\text{trip}}(K\text{FreeSq}, n)$
Extremal rays

For graded modules or squarefree modules, the extremal rays for the cone of Betti diagrams, are given by Betti diagrams of pure resolutions:

\[ S(-d_0)^{\beta_0,d_0} \leftarrow S(-d_1)^{\beta_1,d_1} \leftarrow \cdots \leftarrow S(-d_p)^{\beta_p,d_p} \]

where \( 0 \leq p \leq n \).
Conjecture on extremal rays

Homological triplets

Conjecture (Totally pure complexes)

In $C^{\text{trip}}(K\text{FreeSq}, n)$ the extremal rays are given by triplets $(H, B, C)$ of pure free squarefree complexes

$$F_\bullet : S(-d_0)^{\beta_0,d_0} \leftarrow S(-d_1)^{\beta_1,d_1} \leftarrow \cdots \leftarrow S(-d_p)^{\beta_p,d_p}$$

such that:

1. For every $p < q$ with $H_p(F_\bullet)$ and $H_q(F_\bullet)$ nonzero and $H_i(F_\bullet) = 0$ when $p < i < q$, then:
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$$\min\{d \mid H_{q,d} \neq 0\} - \text{Krulldim}H_p \geq q - p + 1.$$ 

II. Similarly for the cohomology $C$. 

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(Note. This is also meaningful for $C^{\text{trip}}(K\text{Freemod}, n)$.)
Dualities

\[ A : \text{ Alexander duality on Sq. } \]
\[ D : \text{ standard duality on FreeSq: } D = \text{Hom}_S(-, \omega_S). \]
Dualities

\( \mathbb{A} \) : Alexander duality on \( \text{Sq} \).

\( \mathbb{D} \) : standard duality on \( \text{Free} \text{Sq} \):
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\mathbb{D} = \text{Hom}_S(\_\_ , \omega_S).
\]

Then

- I. \( \Leftrightarrow \mathbb{A} \circ \mathbb{D}(F_\bullet) \) being a pure free complex.
- II: \( \Leftrightarrow (\mathbb{A} \circ \mathbb{D})^2(F_\bullet) \) being a pure free complex.
- Note that \( (\mathbb{A} \circ \mathbb{D})^3 \cong \text{Id}[-n] \).
The basic problems

**Problem 1.** Show existence of totally pure complexes.
**Problem 2.** Show they are exactly the extremal rays.
Connection between cones
A surprising connection

Theorem

There is an injection of cones

\[ C^{cohom}(Kcoh_P, n) \overset{\iota}{\hookrightarrow} C^{trip}(KFreeSq, n). \]

Moreover this injection comes from an algebraic association

\[ \mathcal{F}^\bullet \overset{\hat{i}}{\mapsto} W(\mathcal{F}^\bullet), \text{ a free squarefree complex} \]
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**Conjecture**

The map \( \iota \) is an isomorphism of cones.
Problem 1 on the existence of totally pure complexes can then be transferred to a problem on the existence of certain complexes of coherent sheaves.
Existence I

Theorem

For each numerical triple \((H, B, C)\) where \(H_p\) is nonzero only for one \(p\), and \(C_p\) is nonzero for only one \(p\), this totally pure free squarefree complex exists.

It is in fact nothing but a pure free resolution of a CM squarefree module.
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In characteristic zero they are the image by \(\hat{\iota}\) of natural \(GL(W)\)-equivariant vector bundles on projective space \(\mathbb{P}(W)\).
Restriction to vector bundles

Theorem (Essentially Eisenbud-Schreyer)

The restriction

\[ C^{\text{cohom}}(\text{vect}_{\mathbb{P}^c}, n) \twoheadrightarrow C^{\text{trip}}(CMSq^c, n) \]
Theorem (Essentially Eisenbud-Schreyer)

The restriction

\[ C^{\text{coh}}(\text{vect}_{\mathbb{P}^c}, n) \cong C^{\text{betti}}(CMSq^c, n) \]

is an isomorphism of cones.
Existence goal

Show that for all possible *numerical* triplets \((H, B, C)\) there does exist a free squarefree complex with these homological invariants.

Show that such complexes are in the image of \(\hat{\iota}\).
Existence II

**Theorem (F.-S.Sam)**

For all numerically possible \((H, B, C)\) with one nonzero homology module \(H_p\), and with two nonzero cohomology modules \(C_q\), there does exist a totally pure free squarefree complex.

The coherent sheaves which by \(\hat{i}\) map to these complexes arise as equivariant coherent sheaves for a maximal parabolic subgroup \(P \subseteq GL(W)\).
Conclusion

Characterize the following cones, i.e. their extremal rays:

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- The cones 1. and 3. are likely isomorphic.
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- Even if all totally pure complexes exists it is still only conjectural that they are all the extremal rays.
- Cone 2. is still uncharted, but various projections of subcones have been understood.