Toric ideals finitely generated up to symmetry

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based on

(with Jan Draisma, Rob H. Eggermont, Robert Krone)

and

[arXiv:1401.0397] Equivariant lattice generators and Markov bases
(with Thomas Kahle, Robert Krone)
Equivariant Gröbner bases (EGBs)

Ideals in some $\infty$-dimensional rings can be represented finitely:

- $K[x_1, x_2, \ldots]$ with the action of $\mathcal{S}_\infty$ is Noetherian up to symmetry: every equivariant ideal is generated by orbits of finitely many elements.
- ... still true if $\mathcal{S}_\infty$ is replaced with the monoid

$$\text{Inc}(\mathbb{N}) = \{\text{monotonous maps } \mathbb{N} \to \mathbb{N}\}.$$  

- E.g., $\langle x_1, x_2, \ldots \rangle = \langle x_{2014} \rangle_{\mathcal{S}_\infty} = \langle x_1 \rangle_{\text{Inc}(\mathbb{N})}$.
- There is an algorithm to compute an equivariant GB.

... or perhaps are represented finitely:

- $K[y_{i,j} \mid i, j \in \mathbb{N}]$ and $K[y_{\{i,j\}} \mid i, j \in \mathbb{N}]$ with the diagonal action of $\mathcal{S}_\infty$ are not equivariantly Noetherian.
- However, if EGB algorithm terminates, then the output is an EGB.
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Equivariant maps

General goal: study families with $\mathcal{S}_n$ symmetry as $n \to \infty$.

Let $K[Y]$ and $K[X]$ be polynomial rings with $\mathcal{S}_\infty$-actions.

- A map $\phi : K[Y] \to K[X]$ is a $\mathcal{S}_\infty$-equivariant map if
  \[ \sigma \phi(f) = \phi(\sigma f) \quad \text{for all } \sigma \in \mathcal{S}_\infty, f \in K[Y]. \]

- An ideal $I \subset K[Y]$ is a $\mathcal{S}_\infty$-invariant ideal if
  \[ \sigma I \subseteq I \quad \text{for all } \sigma \in \mathcal{S}_\infty. \]

- $\phi$ is $\mathcal{S}_\infty$-equivariant $\implies$ ker $\phi$ is a $\mathcal{S}_\infty$-invariant ideal.
Example (Two theorems)

[de Loera-Sturmfels-Thomas] Let $\phi : y_{i,j} \mapsto x_i x_j$ for $i \neq j \in \mathbb{N}$,

$$\ker \phi = \langle y_{1,2} y_{3,4} - y_{1,4} y_{3,2} \rangle_{\mathcal{S}_\infty}.$$

[Aoki-Takemura] Let $\phi : y_{i,j} \mapsto x_i x_j$ for $i \neq j \in \mathbb{N}$,

$$\ker \phi = \langle y_{12} y_{34} - y_{14} y_{32}, y_{12} y_{23} y_{31} - y_{21} y_{32} y_{13} \rangle_{\mathcal{S}_\infty}.$$

Idea of a proof: eliminate (using EGB) in the ring $K[x_i, y_{i,j} \mid i, j \in \mathbb{N}]$.

Definition: Equivariant Markov basis of $\phi$ is a set generating $\ker \phi$ up to symmetry.
Kernel of $\phi : y_{ij} \mapsto x_i^2 x_j$

Using EGB for elimination get a Markov basis:

$$y_{1,1} y_{0,2} - y_{1,2} y_{0,3}$$
$$y_{2,0} y_{1,0} - y_{1,2} y_{0,2}$$
$$y_{2,1} y_{0,1} - y_{1,2} y_{0,2}$$
$$y_{2,3} y_{0,1} - y_{2,1} y_{0,3}$$
$$y_{2,3} y_{1,0} - y_{2,0} y_{1,3}$$
$$y_{3,1} y_{2,0} - y_{3,0} y_{2,1}$$
$$y_{3,2} y_{0,1} - y_{3,1} y_{0,2}$$
$$y_{3,2} y_{1,0} - y_{3,0} y_{1,2}$$

$$y_{1,2} y_{0,1}^2 - y_{1,0} y_{0,2}$$
$$y_{2,0} y_{0,1}^2 - y_{1,0} y_{0,2}$$
$$y_{2,1} y_{0,2} - y_{2,0} y_{0,1}$$
$$y_{2,1} y_{1,0} y_{0,2} - y_{2,0} y_{1,2} y_{0,1}$$
$$y_{2,1} y_{1,0} y_{0,2} - y_{2,0} y_{1,2} y_{0,1}$$
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$$y_{2,1} y_{1,0}^2 - y_{2,0} y_{1,2}$$
$$y_{2,1} y_{0,1} y_{0,3} - y_{2,0} y_{1,3} y_{0,1}$$
$$y_{2,1} y_{0,1} y_{0,3} - y_{2,0} y_{1,3} y_{0,1}$$

$$y_{2,3} y_{1,2} y_{0,2} - y_{2,0} y_{1,3} y_{0,1}$$

51 generators, width 6, degree 5.

(Computation in 4ti2 by Draisma, EquivariantGB in M2 by Krone)
Questions

- Does a finite equivariant Markov basis (or lattice generating set) exist?
- Can we compute one?
- Can we find small bases?
  - **Size**: number of generators.
  - **Degree**: maximum degree of the generators.
  - **Width**: largest index value used by the generators.

Can we answer these for the one-monomial map

\[ \phi : K[y_{ij} \mid i \neq j \in \mathbb{N}] \to K[x_1, x_2, \ldots] \]

\[ y_{ij} \mapsto x_i^a x_j^b \]

where \(a > b\) are coprime?
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where \( a > b \) are coprime?
Factor $\phi$ as

$$\phi : K[Y] \xrightarrow{\pi} K \left[ \frac{z_{11}, z_{12}, \ldots}{z_{21}, z_{22}, \ldots} \right] \xrightarrow{\psi} K[x_1, x_2, \ldots]$$

$$\pi : y_{ij} \mapsto z_{1i}z_{2j}$$

$$\psi : z_{1i} \mapsto x^a_i$$

$$\psi : z_{2i} \mapsto x^b_i$$

- $\ker \phi = \ker \pi + \pi^{-1}(\text{im } \pi \cap \ker \psi)$.
- $\ker \pi = \langle y_{12}y_{34} - y_{14}y_{32}, y_{12}y_{23}y_{31} - y_{21}y_{32}y_{13} \rangle S_\infty$.
- $(\ker \pi)K[Y^\pm] = \langle y_{12}y_{34} - y_{14}y_{32} \rangle S_\infty$.
- What does $\text{im } \pi \cap \ker \psi$ look like?
Equivariant GB
One-monomial map
General momonial map

Lattice picture

\[ A_\psi : M_{2 \times N} \to M_{1 \times N} \]
\[
\begin{bmatrix}
  c_{11} & c_{12} & \cdots \\
  c_{21} & c_{22} & \cdots
\end{bmatrix}
\mapsto
\begin{bmatrix}
  ac_{11} + bc_{21} & ac_{12} + bc_{22} & \cdots
\end{bmatrix}.
\]

- \( A_\psi \) multiplies matrices by \([a \ b]\) on the left.
- Any \( v \in \ker A_\psi \) must have all columns in the span of \( \begin{bmatrix} b \\ -a \end{bmatrix} \).
- **Lemma:** \( \text{im} \ \pi \cap \ker \psi \) is generated by \( \mathcal{G}_\infty \)-orbits of \( z^{C_1} - z^{C_2} \) such that
  \[
  C_1 - C_2 = \begin{bmatrix}
  b & -b & 0 & \cdots \\
  -a & a & 0 & \cdots
\end{bmatrix}.
  \]
- \( z^{b}_{11} z^a_{22} - z^{b}_{12} z^a_{21} \) generates the lattice.
im $\pi \cap \ker \psi$ is generated by $\mathfrak{S}_\infty$-orbits of $z^{C_1} = z^{C_2}$ with the following $C_1, C_2$

(i) For each $0 \leq n \leq a - b$,

\[
C_1 = \begin{bmatrix}
b + n & n & c_{13} & c_{14} & \cdots \\
0 & a & c_{23} & c_{24} & \cdots
\end{bmatrix}, \quad C_2 = \begin{bmatrix}
n & b + n & c_{13} & c_{14} & \cdots \\
a & 0 & c_{23} & c_{24} & \cdots
\end{bmatrix}
\]

where $\sum_{j \geq 3} c_{1j} = a - b - n$ and $\sum_{j \geq 3} c_{2j} = n$.

(ii) For each $1 \leq n \leq b$,

\[
C_1 = \begin{bmatrix}
b & 0 & a - b + n & 0 & \cdots \\
0 & a & n & 0 & \cdots
\end{bmatrix}, \quad C_2 = \begin{bmatrix}
0 & b & a - b + n & 0 & \cdots \\
a & 0 & n & 0 & \cdots
\end{bmatrix}.
\]

(Experiments using $4ti2$ and $Macaulay2$ helped.)
Markov basis formula [Kahle-Krone-L.]

A $\mathcal{G}_\infty$-equivariant Markov basis for $\phi$ is

(i) $y_{12}y_{34} - y_{14}y_{32}$;

(ii) $y_{12}y_{23}y_{31} - y_{21}y_{32}y_{13}$;

(iii) for each $0 \leq n \leq a - b$,

$$
y_{12}^{b+n} \prod_{j \geq 3} y_{j2}^{c_{1j}} y_{2j}^{c_{2j}} - y_{21}^{b+n} \prod_{j \geq 3} y_{j1}^{c_{1j}} y_{1j}^{c_{2j}}
$$

where $\sum_{j \geq 3} c_{1j} = a - b - n$ and $\sum_{j \geq 3} c_{2j} = n$;

(iv) for each $1 \leq n \leq b$,

$$
y_{12}^{b-n} y_{13} y_{32}^{a-b+n} - y_{21}^{b-n} y_{23} y_{31}^{a-b+n}.
$$

Size = $O((a - b)^{(a-b)})$,  \hspace{2cm} (for $(a, b) = (2, 1)$)

degree = $\max(a + b, 2a - b)$,  \hspace{2cm} (5)

width = $\max(4, a - b + 2)$.  \hspace{2cm} (3)
\( S_{\infty} \)-equivariant lattice generating set

For width = 2, \( \phi : y_{ij} \mapsto x_i^a x_j^b \),

- [Kahle, Krone, L.] Up to symmetry, the lattice generators are
  - \( y_{12} y_{34} - y_{14} y_{32} \);
  - \( y_{21} y_{31}^{a-b} - y_{12} y_{32}^{a-b} \).
- Note: Size = 2, degree = \( a \), width = 4. These are small!

In general,

\[
\phi : y(i_1, \ldots, i_k) \mapsto x_{i_1}^{a_1} \cdots x_{i_k}^{a_k}.
\]

- [Hillar-delCampo] The lattice (equivariant binomial ideal) can be generated in width \( 2d - 1 \), where \( d = \sum a_i \).
- [Kahle-Krone-L.] There is a formula for lattice generators with
  - width \( \leq k + 2 \) and \( (k^2 + k - 2)/2 \) elements or
  - width \( 2k \) and \( k - 1 \) elements,
- Markov bases seem involved for \( k \geq 3 \).
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- Note: Size = 2, degree = \(a\), width = 4. These are small!

In general,

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Theorem (Draisma, Eggermont, Krone, L.)

Any $\mathfrak{S}_\infty$-equivariant toric map

$$\phi : K[Y] \to K \begin{bmatrix} x_{11}, x_{12}, & \cdots \\ \vdots \\ x_{k1}, x_{k2}, & \cdots \end{bmatrix}$$

where $Y$ has a finite number of $\mathfrak{S}_\infty$ orbits, has a finite $\mathfrak{S}_\infty$-equivariant Markov basis.

- There exists an algorithm to construct a Markov basis above... (not implemented!)
- Implies [de Loera-Sturmfels-Thomas], [Aoki-Takemura], and ...

[Hillar-Sullivant] Let $E$ be a hypergraph on $[k]$ and let

$$\phi_E : y(i_1, \ldots, i_k) \mapsto \prod_{e=(j_1, \ldots, j_s) \in E} x_e(i_{j_1}, \ldots, i_{j_s}).$$

If $\forall e$ has at most one infinite index, $\exists$ a finite equivariant Markov basis.
Factoring the map

Main idea: factor the map $\phi : K[Y] \to K[X]$ as $\phi = \psi \circ \pi$.

- Let $Y = \{y_{p,J} \mid p, J\}$, where
  - $p \in [N]$, $N$ is the number of $\mathcal{S}_\infty$-orbits;
  - $J$ is a $k_p$-tuple of distinct natural numbers.

- $\pi : K[Y] \to K[Z]$ is given by
  $$y_{p,J} \mapsto \prod_{l \in [k_p]} z_{p,l,j_l}.$$

- $\psi$ ... captures the rest.

- Want: show that $\ker \pi$ is finitely generated and $\text{im} \pi$ is Noetherian up to symmetry.
Matching monoid

Generate a monoid by $\pi(y_p, J) = z^A$, where $A \in \prod_{p \in [N]} \mathbb{N}^{[k_p] \times \mathbb{N}}$ is an $[N]$-tuple of finite-by-$\infty$ matrices $A_p$. 
• **Proposition:** For an $N$-tuple $A \in \prod_{p \in [N]} \mathbb{N}^{[k_p] \times \mathbb{N}}$, $z^A \in \text{im } \pi$ iff $\forall p \in [N]$ the matrix $A_p \in \mathbb{N}^{[k_p] \times \mathbb{N}}$ has
  • all row sums equal to a number $d_p \in \mathbb{N}$ and
  • all column sums $\leq d_p$.

We call such $A$ good.

• ...

• ...

• ...

• **Proposition:** The $\text{Inc}(\mathbb{N})$ divisibility order on the matching monoid (of good $A$) is a wpo.

• This settles the Noetherianness of $\text{im } \pi$. 
Putting things together

- **Proposition**: \( \ker \pi \) is generated by binomials in \( y_p, J \) of degree at most \( 2 \max_p k_p - 1 \).
- **Recall**: \( \ker \phi = \ker \pi + \pi^{-1}(\text{im} \, \pi \cap \ker \psi) \). Therefore, \( \ker \phi \) is finitely generated.

- It is possible to generalize **Buchberger’s algorithm** to \( K[M] \) for a monoid \( M \) with a wpo... if all ingredients are made effective.
- One can find an equivariant GB for \( \ker \pi \) of the same degree as in Proposition above (w.r.t. a certain monomial order).
- There exists an algorithm to construct a Markov basis for \( \ker \phi \).
Summary

- Computing equivariant Markov bases is hard for machines.
- It is possible to find relatively small bases for some families.
- [DEKL] main theorem implies Noetherianness up to symmetry for the kernel of a monomial map in a large class of maps with the image in a $\mathfrak{S}_\infty$-Noetherian ring.
- What parts of commutative algebra transplant to the $\infty$-dimensional $\mathfrak{S}_\infty$-equivariant setting?