Toric ideals finitely generated up to symmetry

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based on
[arXiv:1306.0828] Noetherianity for infinite-dimensional toric varieties
(with Jan Draisma, Rob H. Eggermont, Robert Krone)
and
[arXiv:1401.0397] Equivariant lattice generators and Markov bases
(with Thomas Kahle, Robert Krone)

Equivariant Gröbner bases (EGBs)

Ideals in some ∞-dimensional rings can be represented finitely:

- $K[x_1, x_2, \ldots]$ with the action of $\mathfrak{S}_\infty$ is Noetherian up to symmetry: every equivariant ideal is generated by orbits of finitely many elements.
- ... still true if $\mathfrak{S}_\infty$ is replaced with the monoid

$$\mathrm{Inc}(\mathbb{N}) = \{\text{monotonous maps } \mathbb{N} \to \mathbb{N}\}.$$

- E.g., $\langle x_1, x_2, \ldots \rangle = \langle x_{2014} \rangle_{\mathfrak{S}_\infty} = \langle x_1 \rangle_{\mathrm{Inc}(\mathbb{N})}$.
- There is an algorithm to compute an equivariant GB.

... or perhaps are represented finitely:

- $K[y_{ij} \mid i, j \in \mathbb{N}]$ and $K[y_{\{i,j\}} \mid i, j \in \mathbb{N}]$ with the diagonal action of $\mathfrak{S}_\infty$ are not equivariantly Noetherian.
- However, if EGB algorithm terminates, then the output is an EGB.

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Equivariant maps

General goal: study families with $\mathfrak{S}_n$ symmetry as $n \to \infty$.

Let $K[Y]$ and $K[X]$ be polynomial rings with $\mathfrak{S}_\infty$-actions.

- A map $\phi : K[Y] \to K[X]$ is a $\mathfrak{S}_\infty$-equivariant map if

$$\sigma\phi(f) = \phi(\sigma f) \quad \text{for all } \sigma \in \mathfrak{S}_\infty, f \in K[Y].$$

- An ideal $I \subset K[Y]$ is a $\mathfrak{S}_\infty$-invariant ideal if

$$\sigma I \subseteq I \quad \text{for all } \sigma \in \mathfrak{S}_\infty.$$

- ϕ is $\mathfrak{S}_\infty$-equivariant $\Rightarrow \ker\phi$ is a $\mathfrak{S}_\infty$-invariant ideal.

Example (Two theorems)

[de Loera-Sturmfels-Thomas] Let $\phi : y_{\{i,j\}} \mapsto x_i x_j$ for $i \neq j \in \mathbb{N}$,

$$\ker \phi = \langle y_{\{1,2\}} y_{\{3,4\}} - y_{\{1,4\}} y_{\{3,2\}} \rangle_{\mathfrak{S}_\infty}.$$

[Aoki-Takemura] Let $\phi : y_{ij} \mapsto x_i x_j$ for $i \neq j \in \mathbb{N}$,

$$\ker \phi = \langle y_{12} y_{34} - y_{14} y_{32},\ y_{12} y_{23} y_{31} - y_{21} y_{32} y_{13} \rangle_{\mathfrak{S}_\infty}.$$

Idea of a proof: eliminate (using EGB) in the ring $K[x_i, y_{ij} \mid i, j \in \mathbb{N}]$.

Definition: Equivariant Markov basis of ϕ is a set generating $\ker \phi$ up to symmetry.

Kernel of $\phi : y_{ij} \mapsto x_i^2 x_j$

Using EGB for elimination get a Markov basis:

$$
\begin{aligned}
y_{1,3}y_{0,2} - y_{1,2}y_{0,3}\\
y_{2,0}y_{1,0} - y_{1,2}y_{0,2}\\
y_{2,1}y_{0,1} - y_{1,2}y_{0,2}\\
y_{2,3}y_{0,1} - y_{2,1}y_{0,3}\\
y_{2,3}y_{1,0} - y_{2,0}y_{1,3}\\
y_{3,1}y_{2,0} - y_{3,0}y_{2,1}\\
y_{3,2}y_{0,1} - y_{3,1}y_{0,2}\\
y_{3,2}y_{1,0} - y_{3,0}y_{1,2}
\end{aligned}
$$

$$
\begin{aligned}
y_{1,2}y_{0,1}^2 - y_{1,0}^2y_{0,2}\\
y_{2,0}y_{0,1}^2 - y_{1,0}y_{0,2}^2\\
y_{2,1}y_{0,2}^2 - y_{2,0}^2y_{0,1}\\
y_{2,1}y_{1,0}y_{0,2} - y_{2,0}y_{1,2}y_{0,1}\\
y_{2,1}y_{1,0}^2 - y_{1,2}^2y_{0,1}\\
y_{2,1}^2y_{0,2} - y_{2,0}^2y_{1,2}\\
y_{2,1}^2y_{1,0} - y_{2,0}y_{1,2}^2\\
y_{2,1}y_{1,0}y_{0,3} - y_{2,0}y_{1,3}y_{0,1}\\
y_{2,1}^2y_{0,3} - y_{2,0}^2y_{1,3}\\
y_{2,3}y_{1,2}y_{0,2} - y_{2,0}^2y_{1,3}
\end{aligned}
$$

$$
\begin{aligned}
y_{3,0}y_{1,2}y_{0,2} - y_{2,0}y_{1,3}y_{0,3}\\
y_{3,0}y_{1,2}^2 - y_{2,0}y_{1,3}^2\\
y_{3,0}y_{2,1}^2 - y_{2,3}y_{2,0}y_{1,3}\\
y_{3,1}y_{0,2}^2 - y_{2,1}y_{0,3}^2\\
y_{3,1}y_{1,0}y_{0,2} - y_{3,0}y_{1,2}y_{0,1}\\
y_{3,1}y_{1,2}y_{0,2} - y_{2,1}y_{1,3}y_{0,3}\\
y_{3,1}y_{2,3}y_{0,3} - y_{3,0}^2y_{2,1}\\
y_{3,1}^2y_{0,2} - y_{3,0}^2y_{1,2}\\
y_{3,2}y_{1,3}y_{0,3} - y_{3,0}^2y_{1,2}\\
y_{3,2}y_{2,0}y_{1,3} - y_{3,0}y_{2,3}y_{1,2}\\
y_{3,2}y_{2,0}y_{1,4} - y_{3,0}y_{2,4}y_{1,2}\\
y_{3,2}y_{2,1}y_{0,3} - y_{3,1}y_{2,3}y_{0,2}\\
y_{3,2}y_{2,1}y_{0,4} - y_{3,1}y_{2,4}y_{0,2}\\
y_{4,0}y_{2,3}y_{1,3} - y_{3,0}y_{2,4}y_{1,4}\\
y_{4,1}y_{2,3}y_{0,3} - y_{3,1}y_{2,4}y_{0,4}\\
y_{4,2}y_{1,3}y_{0,3} - y_{3,2}y_{1,4}y_{0,4}\\
y_{4,2}y_{2,0}y_{1,3} - y_{4,0}y_{2,3}y_{1,2}\\
y_{4,2}y_{2,1}y_{0,3} - y_{4,1}y_{2,3}y_{0,2}
\end{aligned}
$$

$$
\begin{aligned}
y_{2,1}y_{1,2}y_{0,3}y_{0,2} - y_{2,0}^2y_{1,3}y_{0,1}\\
y_{3,1}y_{2,3}y_{1,3}y_{0,4} - y_{3,0}^2y_{2,1}y_{1,4}\\
y_{3,1}y_{2,3}^2y_{0,4} - y_{3,0}^2y_{2,4}y_{2,1}\\
y_{3,2}y_{2,3}y_{1,3}y_{0,4} - y_{3,0}^2y_{2,4}y_{1,2}\\
y_{4,1}y_{2,3}y_{1,4}y_{0,4} - y_{4,0}^2y_{2,1}y_{1,3}\\
y_{4,1}y_{3,2}y_{1,4}y_{0,4} - y_{4,0}^2y_{3,1}y_{1,2}\\
y_{4,1}y_{3,4}y_{2,4}y_{0,5} - y_{4,0}^2y_{3,1}y_{2,5}
\end{aligned}
$$

$$
\begin{aligned}
y_{2,1}y_{1,2}^2y_{0,3}^2 - y_{2,0}^2y_{1,3}^2y_{0,1}\\
y_{2,1}y_{1,2}^2y_{0,4}y_{0,3} - y_{2,0}^2y_{1,4}y_{1,3}y_{0,1}\\
y_{3,2}y_{2,3}^2y_{1,4}y_{0,4} - y_{3,0}^2y_{2,4}^2y_{1,2}\\
y_{3,2}y_{2,3}^2y_{1,4}y_{0,5} - y_{3,0}^2y_{2,5}y_{2,4}y_{1,2}\\
y_{4,1}y_{2,3}y_{1,4}^2y_{0,5} - y_{4,0}^2y_{2,1}y_{1,5}y_{1,3}\\
y_{4,1}y_{3,2}y_{1,4}^2y_{0,5} - y_{4,0}^2y_{3,1}y_{1,5}y_{1,2}\\
y_{4,3}y_{4,0}^2y_{3,2}y_{3,1} - y_{4,2}y_{4,1}y_{3,4}^2y_{0,3}\\
y_{5,1}y_{4,2}y_{3,5}^2y_{0,3} - y_{5,0}^2y_{4,3}y_{3,2}y_{3,1}
\end{aligned}
$$

51 generators, width 6, degree 5.
(Computation in 4ti2 by Draisma, EquivariantGB in M2 by Krone)

Questions

- Does a finite equivariant Markov basis (or lattice generating set) exist?
- Can we compute one?
- Can we find small bases?
  - Size: number of generators.
  - Degree: maximum degree of the generators.
  - Width: largest index value used by the generators.

Can we answer these for the one-monomial map

$$\phi : K[y_{ij} \mid i \neq j \in \mathbb{N}] \to K[x_1, x_2, \ldots]$$

$$y_{ij} \mapsto x_i^a x_j^b$$

where $a > b$ are coprime?

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Factor ϕ as

$$\phi : K[Y] \xrightarrow{\pi} K\begin{bmatrix} z_{11}, z_{12}, \ldots \\ z_{21}, z_{22}, \ldots \end{bmatrix} \xrightarrow{\psi} K[x_1, x_2, \ldots]$$

$$\pi : y_{ij} \mapsto z_{1i} z_{2j}$$

$$\psi : z_{1i} \mapsto x_i^a$$

$$\psi : z_{2i} \mapsto x_i^b$$

- $\ker \phi = \ker \pi + \pi^{-1}(\operatorname{im} \pi \cap \ker \psi)$.
- $\ker \pi = \langle y_{12}y_{34} - y_{14}y_{32},\ y_{12}y_{23}y_{31} - y_{21}y_{32}y_{13} \rangle \mathfrak{S}_\infty$.
- $(\ker \pi) K[Y^{\pm}] = \langle y_{12}y_{34} - y_{14}y_{32} \rangle \mathfrak{S}_\infty$.
- What does $\operatorname{im} \pi \cap \ker \psi$ look like?

Lattice picture

$$A_\psi : M_{2\times\mathbb{N}} \to M_{1\times\mathbb{N}}$$

$$\begin{bmatrix} c_{11} & c_{12} & \cdots \\ c_{21} & c_{22} & \cdots \end{bmatrix} \mapsto \begin{bmatrix} ac_{11}+bc_{21} & ac_{12}+bc_{22} & \cdots \end{bmatrix}.$$

- A_ψ multiplies matrices by $[a\ b]$ on the left.
- Any $v \in \ker A_\psi$ must have all columns in the span of $\begin{bmatrix} b \\ -a \end{bmatrix}$.
- Lemma: $\operatorname{im}\pi \cap \ker\psi$ is generated by $\mathfrak{S}_\infty$-orbits of $z^{C_1} - z^{C_2}$ such that
$$C_1 - C_2 = \begin{bmatrix} b & -b & 0 & \cdots \\ -a & a & 0 & \cdots \end{bmatrix}.$$
- $z_{11}^b z_{22}^a - z_{12}^b z_{21}^a$ generates the lattice.

$\operatorname{im}\pi \cap \ker\psi$ is generated by $\mathfrak{S}_\infty$-orbits of $z^{C_1} - z^{C_2}$ with the following C_1, C_2

(i) For each $0 \le n \le a-b$,

$$C_1 = \begin{bmatrix} b+n & n & c_{13} & c_{14} & \cdots \\ 0 & a & c_{23} & c_{24} & \cdots \end{bmatrix}, \; C_2 = \begin{bmatrix} n & b+n & c_{13} & c_{14} & \cdots \\ a & 0 & c_{23} & c_{24} & \cdots \end{bmatrix}$$

where $\sum_{j\ge 3} c_{1j} = a-b-n$ and $\sum_{j\ge 3} c_{2j} = n$.

(ii) For each $1 \le n \le b$,

$$C_1 = \begin{bmatrix} b & 0 & a-b+n & 0 & \cdots \\ 0 & a & n & 0 & \cdots \end{bmatrix}, \; C_2 = \begin{bmatrix} 0 & b & a-b+n & 0 & \cdots \\ a & 0 & n & 0 & \cdots \end{bmatrix}.$$

(Experiments using 4ti2 and Macaulay2 helped.)

Markov basis formula [Kahle-Krone-L.]

A $\mathfrak{S}_\infty$-equivariant Markov basis for ϕ is (for $(a,b)=(2,1)$)

(i) $y_{12}y_{34} - y_{14}y_{32}$; (...)

(ii) $y_{12}y_{23}y_{31} - y_{21}y_{32}y_{13}$; (...)

(iii) for each $0 \le n \le a-b$,

$$y_{12}^{b+n}\prod_{j\ge 3} y_{j2}^{c_{1j}} y_{2j}^{c_{2j}} - y_{21}^{b+n}\prod_{j\ge 3} y_{j1}^{c_{1j}} y_{1j}^{c_{2j}} \qquad \left(\begin{array}{ll} y_{12}y_{32} - y_{21}y_{31}, & n=0; \\ y_{12}^2 y_{23} - y_{21}^2 y_{13}, & n=1; \end{array}\right)$$

where $\sum_{j\ge 3} c_{1j} = a-b-n$ and $\sum_{j\ge 3}^{\infty} c_{2j} = n$;

(iv) for each $1 \le n \le b$,

$$y_{12}^{b-n} y_{13}^{n} y_{32}^{a-b+n} - y_{21}^{b-n} y_{23}^{n} y_{31}^{a-b+n}. \qquad \left(y_{13}y_{32}^2 - y_{23}y_{31}^2. \right)$$

Size = $O((a-b)^{(a-b)})$, (5)
degree = $\max(a+b, 2a-b)$, (3)
width = $\max(4, a-b+2)$. (4)

$\mathfrak{S}_\infty$-equivariant lattice generating set

For width = 2, $\phi : y_{ij} \mapsto x_i^a x_j^b$,

- [Kahle, Krone, L.] Up to symmetry, the lattice generators are
  - $y_{12}y_{34} - y_{14}y_{32}$;
  - $y_{21}^b y_{31}^{a-b} - y_{12}^b y_{32}^{a-b}$.
- Note: Size = 2, degree = a, width = 4. These are small!

In general,

$$\phi : y_{(i_1,\ldots,i_k)} \mapsto x_{i_1}^{a_1} \cdots x_{i_k}^{a_k}.$$

- [Hillar-delCampo] The lattice (equivariant binomial ideal) can be generated in width $2d-1$, where $d = \sum_i a_i$.
- [Kahle-Krone-L.] There is a formula for lattice generators with
  - width $\leq k+2$ and $(k^2+k-2)/2$ elements or
  - width $2k$ and $k-1$ elements,
- Markov bases seem involved for $k \geq 3$.

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Theorem (Draisma, Eggermont, Krone, L.)

Any $\mathfrak{S}_\infty$-equivariant toric map

$$\phi : K[Y] \to K \begin{bmatrix} x_{11}, x_{12}, & \dots \\ \vdots & \\ x_{k1}, x_{k2}, & \dots \end{bmatrix}$$

where Y has a finite number of $\mathfrak{S}_\infty$ orbits, has a finite $\mathfrak{S}_\infty$-equivariant Markov basis.

- There exists an algorithm to construct a Markov basis above... (not implemented!)
- Implies [de Loera-Sturmfels-Thomas], [Aoki-Takemura], and ...

[Hillar-Sullivant] Let E be a hypergraph on $[k]$ and let

$$\phi_E : y_{(i_1,\dots,i_k)} \mapsto \prod_{e=(j_1,\dots,j_s)\in E} x_{e,(i_{j_1},\dots,i_{j_s})}.$$

If $\forall e$ has at most one infinite index, $\exists$ a finite equivariant Markov basis.

Factoring the map

Main idea: factor the map $\phi : K[Y] \to K[X]$ as $\phi = \psi \circ \pi$.

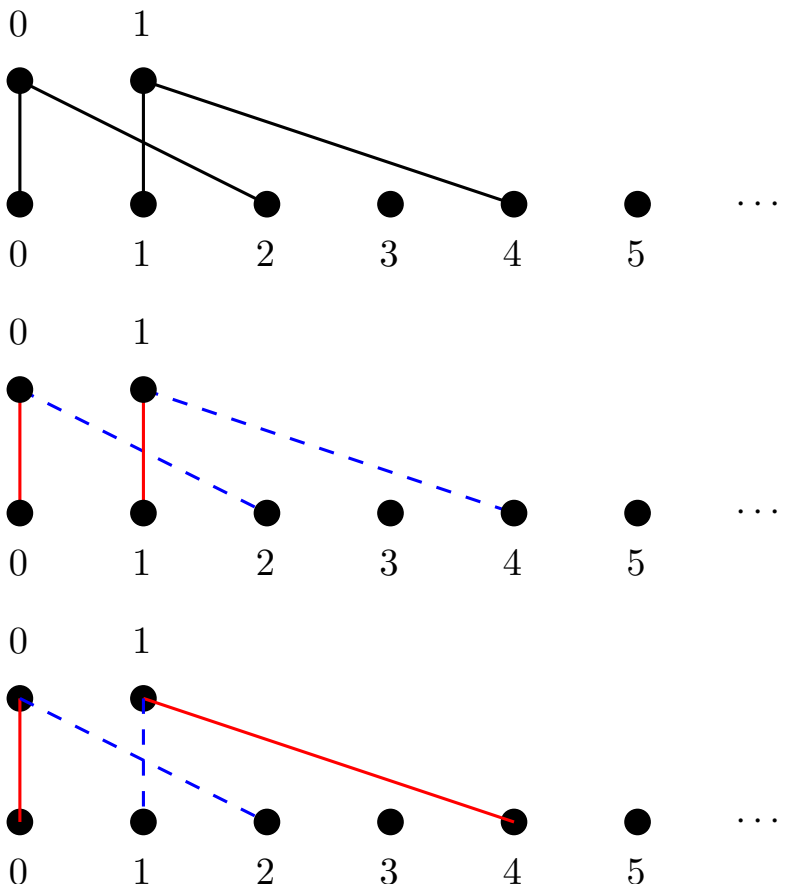
- Let $Y = \{y_{p,J} \mid p, J\}$, where
  - $p \in [N]$, N is the number of $\mathfrak{S}_\infty$-orbits;
  - J is a k_p-tuple of distinct natural numbers.
- $\pi : K[Y] \to K[Z]$ is given by

$$y_{p,J} \mapsto \prod_{l \in [k_p]} z_{p,l,j_l}.$$

- ψ ... captures the rest.
- Want: show that $\ker \pi$ is finitely generated and $\operatorname{im} \pi$ is Noetherian up to symmetry.

Matching monoid

Generate a monoid by $\pi(y_{p,J}) = z^A$, where $A \in \prod_{p\in[N]} \mathbb{N}^{[k_p]\times\mathbb{N}}$ is an $[N]$-tuple of finite-by-∞ matrices A_p.



$$x_p \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 1 & 0 & \cdots \end{bmatrix}$$

$$\|$$

$$\phi(y_{p,(0,1)}y_{p,(2,4)})$$

$$\|$$

$$\phi(y_{p,(0,4)}y_{p,(2,1)})$$

- Proposition: For an N-tuple $A \in \prod_{p\in[N]} \mathbb{N}^{[k_p]\times\mathbb{N}}$,
  $z^A \in \operatorname{im}\pi$ iff $\forall p \in [N]$ the matrix $A_p \in \mathbb{N}^{[k_p]\times\mathbb{N}}$ has
  - all row sums equal to a number $d_p \in \mathbb{N}$ and
  - all column sums $\leq d_p$.

  We call such A good.
- ...
- ...
- ...
- Proposition: The $\mathrm{Inc}(\mathbb{N})$ divisibility order on the matching monoid (of good A) is a wpo.
- This settles the Noetherianness of $\operatorname{im}\pi$.

Putting things together

- Proposition: $\ker \pi$ is generated by binomials in $y_{p,J}$ of degree at most $2 \max_p k_p - 1$.
- Recall: $\ker \phi = \ker \pi + \pi^{-1}(\operatorname{im} \pi \cap \ker \psi)$.
  Therefore, $\ker \phi$ is finitely generated.

- It is possible to generalize Buchberger's algorithm to $K[M]$ for a monoid M with a wpo... if all ingredients are made effective.
- One can find an equivariant GB for $\ker \pi$ of the same degree as in Proposition above (w.r.t. a certain monomial order).
- There exists an algorithm to construct a Markov basis for $\ker \phi$.

Summary

- Computing equivariant Markov bases is hard for machines.
- It is possible to find relatively small bases for some families.
- [DEKL] main theorem implies Noetherianness up to symmetry for the kernel of a monomial map in a large class of maps with the image in a $\mathfrak{S}_\infty$-Noetherian ring.
- What parts of commutative algebra transplant to the ∞-dimensional $\mathfrak{S}_\infty$-equivariant setting?