Markov bases of lattice ideals

Marius Vladoiu

Faculty of Mathematics and Computer Science
University of Bucharest

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Summary

Lattice ideals

Let $L \subset \mathbb{Z}^n$ be a lattice. The lattice ideal $I_L \subset K[x_1, \ldots, x_n]$ is

$$I_L := \langle \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} : \ \mathbf{u} - \mathbf{v} \in L \rangle = \langle \mathbf{x}^{\mathbf{w}^+} - \mathbf{x}^{\mathbf{w}^-} : \ \mathbf{w} \in L \rangle$$

Definition

If L is such that $L \cap \mathbb{N}^n = \{\mathbf{0}\}$ (repectively $L \cap \mathbb{N}^n \neq \{\mathbf{0}\}$) we say that L is *positively graded* (*not positively graded*). Let L_{pure} be the sublattice of L generated by $L \cap \mathbb{N}^n$.

Minimal generating sets of lattice ideals

Definition

A set S is a Markov basis for I_L if S consists of binomials and S is a minimal generating set of I_L of minimal cardinality.

For counting purposes, a binomial B is the same as $-B$.

- How many "different" Markov bases are there?
- Can we compute the cardinality of a Markov basis?
- Can we compute all Markov bases?
- Is there a characteristic shared by different Markov bases?

Degrees and fibers

Let $\mathcal{A}$ be the subsemigroup of $\mathbb{Z}^n/L$ generated by the elements $\{\mathbf{a}_i = \mathbf{e}_i + L : 1 \leq i \leq n\}$, where $\{\mathbf{e}_i : \ 1 \leq i \leq n\}$ is the canonical basis of $\mathbb{Z}^n$ and set

$$\deg_{\mathcal{A}}(\mathbf{x}^{\mathbf{v}}) := v_1\mathbf{a}_1 + \cdots + v_n\mathbf{a}_n \in \mathcal{A}$$

where $\mathbf{x}^{\mathbf{v}} = x_1^{v_1} \cdots x_n^{v_n}$.
It follows that

$$I_L = \langle \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} : \ \deg_{\mathcal{A}}(\mathbf{x}^{\mathbf{u}}) = \deg_{\mathcal{A}}(\mathbf{x}^{\mathbf{v}}) \rangle$$

and that I_L is $\mathcal{A}$-graded.

Definition

Let $\deg_{\mathcal{A}}(\mathbf{x}^{\mathbf{u}}) = \mathbf{b}$. The fiber of $\mathbf{u}$ is the following set of monomials:

$$F_{\mathbf{x}^{\mathbf{u}}} = \deg_{\mathcal{A}}^{-1}(\mathbf{b}) = \{\mathbf{x}^{\mathbf{w}} \mid \deg_{\mathcal{A}}(\mathbf{x}^{\mathbf{w}}) = \mathbf{b}\} = \{\mathbf{x}^{\mathbf{w}} : \ \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{w}} \in I_L\}$$

$L \cap \mathbb{N}^n = \{\mathbf{0}\}$

- The semigroup $\mathcal{A}$ is partially ordered:

$$\mathbf{c} \geq \mathbf{d} \iff \text{ there is } \mathbf{e} \in \mathcal{A} \text{ such that } \mathbf{c} = \mathbf{d} + \mathbf{e} \ .$$

- The $\mathcal{A}$-grading of I_L forces every I_L-fiber to be finite.
- The fibers can be partially ordered by $\deg_{\mathcal{A}}$.
- (the graded Nakayama Lemma "works") All minimal binomial generating sets of I_L have the same cardinality and the same $\mathcal{A}$-degrees.

Generating I_L when $L \cap \mathbb{N}^n = 0$ (CKT 2007)

For every degree $\mathbf{b} \in \mathcal{A}$ define a subideal of I_L generated by the binomials that have $\mathcal{A}$-degrees **less** than $\mathbf{b}$.

Definition

$$I_{L,<\mathbf{b}} = I_{L,<F} = (\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \mid \deg_{\mathcal{A}}(\mathbf{x}^{\mathbf{u}}) = \deg_{\mathcal{A}}(\mathbf{x}^{\mathbf{v}}) \lneqq \mathbf{b}) \subset I_L$$

where F is the fiber at $\mathbf{b}$.

Then we define two graphs.

Definition

First graph Let $G(\mathbf{b})$ be the graph with vertices the elements of the fiber $F = \deg_{\mathcal{A}}^{-1}(\mathbf{b})$ and edges all the sets $\{\mathbf{x}^{\mathbf{u}}, \mathbf{x}^{\mathbf{v}}\}$ whenever $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in I_{L,<\mathbf{b}}$.

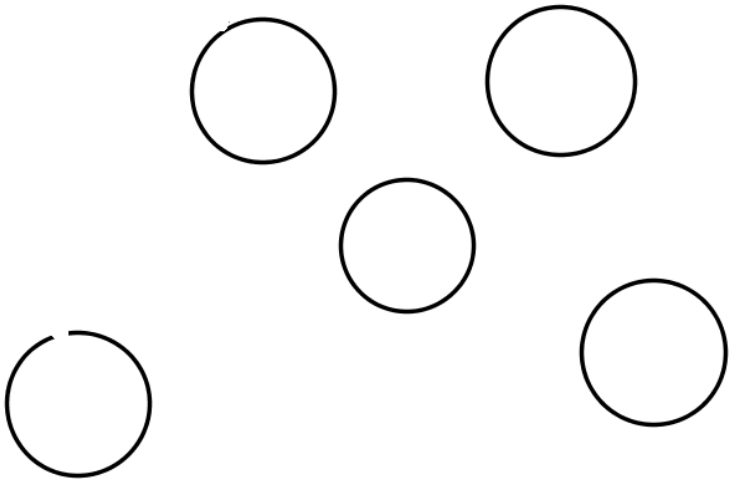
Generating I_L when $L \cap \mathbb{N}^n = 0$ (CKT 2007)

Definition

The Second graph is the complete graph with vertex set the connected components of first graph $G(\mathbf{b})$. Let $T_{\mathbf{b}}$ be a spanning tree of this graph.

For every edge of the tree $T_{\mathbf{b}}$ joining two components of $G(\mathbf{b})$ take one binomial by considering the difference of (two arbitrary) monomials, one from each component. For every $\mathbf{b}$, choose a tree $T_{\mathbf{b}}$ on the graph $G(\mathbf{b})$ (whose vertices are the connected components of the fiber at $\mathbf{b}$) and then choose the binomials. Denote this collection by $\mathcal{F}_{T_{\mathbf{b}}}$.

Picture



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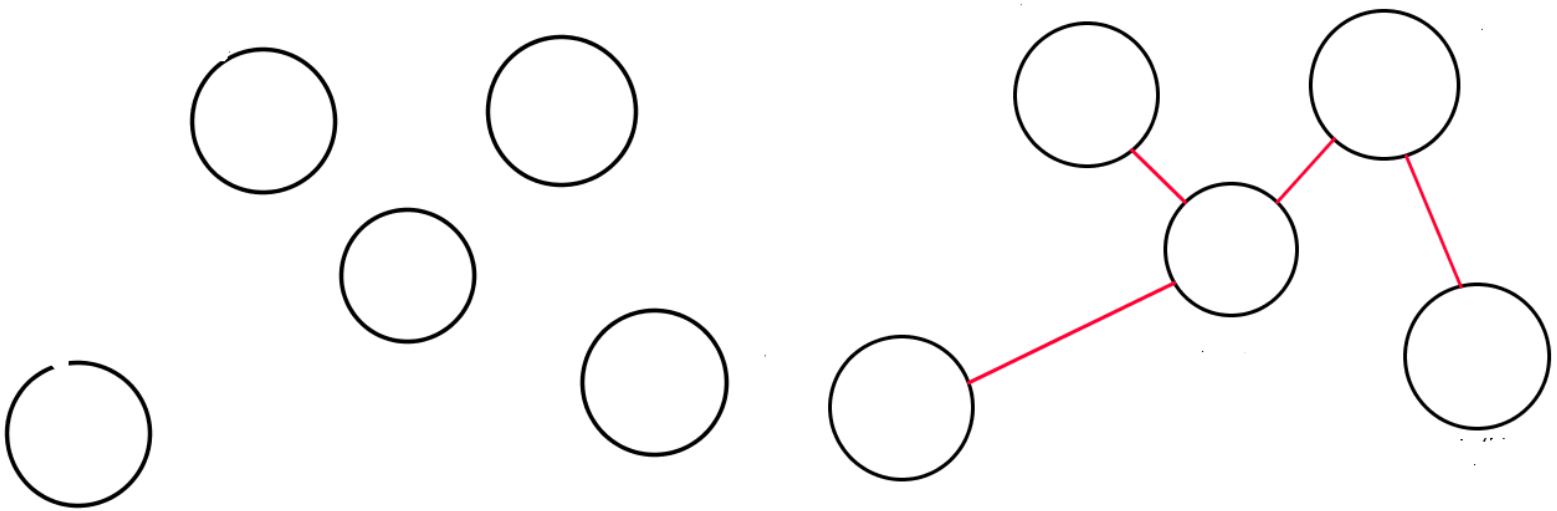

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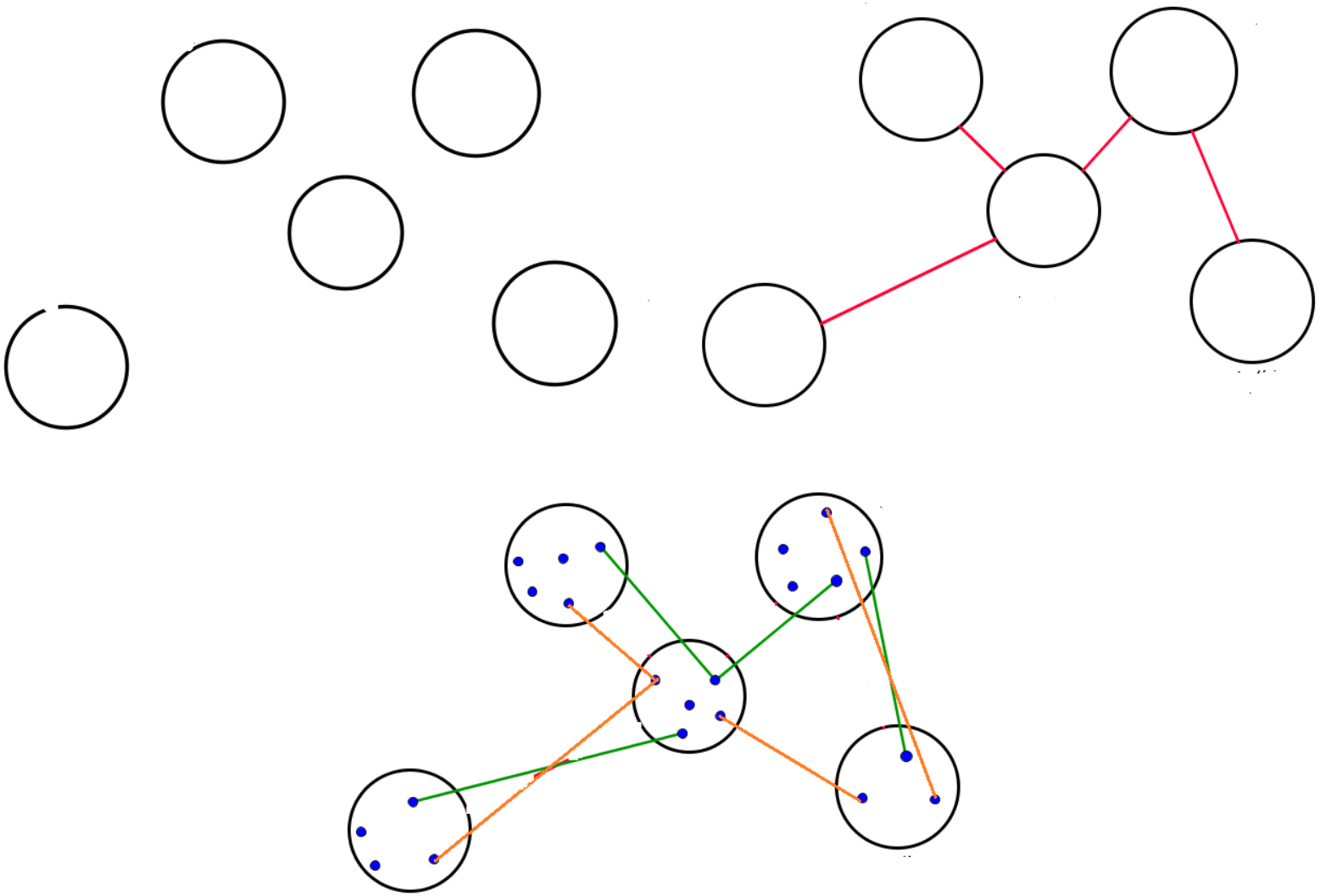

Generating I_L when $L \cap \mathbb{N}^n = 0$ (CKT 2007)

Theorem

The set $\mathcal{F} = \cup_{\mathbf{b} \in \mathcal{A}} \mathcal{F}_{T_{\mathbf{b}}}$ is a Markov basis of I_L.

Let $\mu(I_L)$ be the cardinality of a Markov basis,
$n_{\mathbf{b}}$ the number of connected components of $G(\mathbf{b})$, and
$t_i(\mathbf{b})$ the number of vertices of the ith component.

Theorem

$\mu(I_L) = \sum_{\mathbf{b} \in \mathcal{A}} (n_{\mathbf{b}} - 1)$.

Theorem

The number of different Markov bases of I_L is finite and equal to

$$\prod_{\mathbf{b} \in \mathcal{A}} t_1(\mathbf{b}) \cdots t_{n_{\mathbf{b}}}(\mathbf{b}) (t_1(\mathbf{b}) + \cdots + t_{n_{\mathbf{b}}}(\mathbf{b}))^{n_{\mathbf{b}}-2}.$$

$L \cap \mathbb{N}^n \neq \{\mathbf{0}\}$ (CTV1)

Bad News!

- all fibers are infinite.
- there is no partial order between the fibers.

But

We can consider equivalence classes of fibers under the following equivalence relation:

Definition

$F \equiv_L G \Leftrightarrow (\exists)\mathbf{x}^{\mathbf{u}}, \mathbf{x}^{\mathbf{v}}$ monomials s.t. $\mathbf{x}^{\mathbf{u}}F \subset G$ and $\mathbf{x}^{\mathbf{v}}G \subset F$.

and order the equivalence classes

Definition

Let $\overline{F}$, $\overline{G}$ be two equivalence classes of I_L-fibers. We say that $\overline{F} \leq_{I_L} \overline{G}$ if there exists $\mathbf{x}^{\mathbf{u}}$ such that $\mathbf{x}^{\mathbf{u}}F \subset G$.

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$L \cap \mathbb{N}^n \neq \{\mathbf{0}\}$ (CTV1)

Note that:
1) $L_{pure} = \{\mathbf{0}\}$ implies $\overline{F} = \{F\}$, and the order on the equivalence classes of fibers agrees with the degree-ordering of the fibers.

2) The cardinality of $\overline{F}$ is fixed and is determined by L_{pure}.

3) The Noetherian property of the ring guarantees that all chains of equivalence classes of fibers have a minimal element.

Definition

$$I_{L,<\overline{F}} = (\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \mid \mathbf{x}^{\mathbf{u}}, \mathbf{x}^{\mathbf{v}} \in G, \overline{G} < \overline{F}) \subset I_L.$$

$$I_{L,\leq\overline{F}} = (\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \mid \mathbf{x}^{\mathbf{u}}, \mathbf{x}^{\mathbf{v}} \in G, \overline{G} \leq \overline{F}) \subset I_L.$$

$L \cap \mathbb{N}^n \neq \{\mathbf{0}\}$ (CTV1)

Recall L_{pure}! Denote $\sigma = \mathrm{supp}(L_{pure})$, and $\mathbf{u}^\sigma = (u_i)_{i \notin \sigma}$.

Definition

First graph Let $G(\overline{F})$ be the graph with vertices the elements of $G(M_F)^\sigma$. The edges of $G(\overline{F})$ correspond to binomials of $I_{L,<\overline{F}}$.

Next consider the connected components of $G(\overline{F})$: these are the vertices of the second graph.

Definition

Second graph: The complete graph on the components of $G(\overline{F})$. We call this graph $\Gamma(\overline{F})$.

Consider, as before, spanning trees of $\Gamma(\overline{F})$.

True Generalization

If $L \cap \mathbb{N}^n = \{0\}$ then

- $\sigma = \{\}$
- $\overline{F} = \{F\}$
- $G(M_F)$ is equal to F
- $I_{L,<\overline{F}} = I_{L,<\mathbf{b}}$ where $\mathbf{b}$ is the $\mathcal{A}$-degree of any element in F.

Thus we obtain the same graphs.

Markov bases of pure lattice ideals

Theorem

(CTV1) $B = \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$ *belongs to a Markov basis of* I_L *if and only if* B *is not in* $I_{L,<\overline{F_{\mathbf{x}^{\mathbf{u}}}}}$.

Theorem

(ES95) $I_{L_{pure}}$ *is a complete intersection, generated minimally by* $\mathrm{rank}(L_{pure})$ *elements.*

We complete this theorem by giving a description of all generating sets of $I_{L_{pure}}$ in terms of the exponents of the binomials.

Markov Bases of I_L (CTV1)

Theorem

A set S of binomials of I_L is a Markov basis of I_L if and only if

- *for every $\overline{F}$ the elements of S determine a spanning tree of $\Gamma(\overline{F})$ and*
- *the binomials of S in the equivalence class of the fiber $F_{\mathbf{x}^0}$ minimally generate the lattice generated by $L \cap \mathbb{N}^n$.*

What are the invariants of the Markov bases of I_L?

Theorem

Let $S = \{B_1, \ldots, B_s\}$ be a Markov basis of I_L. The equivalence classes of fibers that correspond to these binomials and their multiplicity in S are uniquely determined and are invariants of I_L.

Markov Bases of I_L (CTV1)

What can we compute?

We can compute the cardinality of a Markov basis, the Markov fibers, the indispensable fibers, the indispensable binomials, and the indispensable monomials.

Theorem

$$\mu(I_L) = r + \sum_{\overline{F} \neq \overline{F}_{\{1\}}} (t(\overline{F}) - 1),$$

where $\mu(I_L)$ is the cardinality of a Markov basis, r is the rank of L_{pure}, and $t(\overline{F})$ is the number of vertices of $\Gamma_{\overline{F}}$.

Markov Bases of I_L (CTV1)

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Binomial Complete Intersection

Definition

Let L be a lattice of rank r. The lattice ideal I_L is called a binomial complete intersection if there exist binomials $B_1, \ldots, B_r$ such that $I_L = \langle B_1, \ldots, B_r \rangle$.

If $L \cap \mathbb{N}^n = \{\mathbf{0}\}$ then complete intersection lattice ideals are automatically binomial complete intersections.

When is the lattice ideal a complete intersection ideal? The problem is completely solved when L is positively graded by a series of articles: Herzog (70), Delorne(76), Stanley (77), Ishida (78), Watanabe(80), Nakajima(85), Schafer (85), Rosales and Garcia-Sanchez (95), Fischer, Morris and Shapiro (95), Scheja, Scheja and Storch (99), Morales and Thoma (05).

Mixed dominating matrices

The final conclusion of this series of articles is that I_L is a complete intersection if and only if the matrix M whose rows correspond to a basis of L is **mixed dominating**.

Definition

A matrix M is mixed dominating if every row of M has a positive and negative entry and M contains no square submatrix with this property.

Example

$$\begin{bmatrix} 1 & -1 & 0 \\ 6 & 0 & -1 \end{bmatrix}$$

Complete intersections and mixed dominating matrices in the general case

We now consider arbitrary sublattices L of $\mathbb{Z}^n$.
In (CTV1) we showed that the rank L is determined by the rank of L^σ (which is a positively graded lattice) and the rank of L_{pure}.
Here L^σ is the sublattice of $(\mathbb{Z}^n)^\sigma$ generated by the vectors $\mathbf{u}^\sigma$.

Theorem

$$\mathrm{rank}(L) = \mathrm{rank}(L^\sigma) + \mathrm{rank}(L_{pure}).$$

Theorem

(CTV1) Let $L \subset \mathbb{Z}^n$ be a lattice. The ideal I_L is binomial complete intersection if and only if there exists a basis of L^σ so that its vectors give the rows of a mixed dominating matrix.

Universal Gröbner basis and Graver basis

Let A be a matrix in $\mathbb{Z}^{m\times n}$. The toric ideal I_A is the lattice ideal $I_{L(A)}$, where $L(A) = \ker_{\mathbb{Z}}(A)$. We assume that $L(A) \cap \mathbb{N}^n = \{\mathbf{0}\}$.

Theorem

(St 95) For any toric ideal I_A the following containments hold:

$$\textit{Universal Gröbner basis of } A \subset \textit{Graver basis of } A$$

What is the relation between the universal Gröbner basis of A and the universal Markov basis of A? What is the relation between the universal Markov basis of A and the Graver basis of A?

Example

Let $I = (x_1x_2 - x_3x_4, x_5x_6 - x_7x_8, x_1^2x_2^2x_3x_4 - x_5x_6x_7x_8)$. This generating set is not part of any reduced Gröbner basis of I.

Example

Let

$$A = \begin{pmatrix} 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 \\ 4 & 0 & 4 & 0 & 3 & 3 & 3 & 3 \\ 4 & 0 & 0 & 4 & 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 & 6 & 0 & 6 & 0 \\ 2 & 2 & 2 & 2 & 6 & 0 & 0 & 6 \end{pmatrix}.$$

It can be shown that

$$I_A = (x_1x_2 - x_3x_4, x_5x_6 - x_7x_8, x_1^2x_2^2x_3x_4 - x_5x_6x_7x_8).$$

The binomial $x_1^2x_2^2x_3x_4 - x_5x_6x_7x_8$ does not belong to a reduced Gröbner basis of I_A since for any monomial order, the initial term of $x_1x_2 - x_3x_4$ divides $x_1^2x_2^2x_3x_4$ while the initial term of $x_5x_6 - x_7x_8$ divides $x_5x_6x_7x_8$.

Markov Polytopes

Theorem

(CKT07, DSS09) $\mathbf{u}$ *is in the universal Markov basis of* A *if and only if* $\mathbf{u}^+$ *and* $\mathbf{u}^-$ *belong to different connected components of* $G_{\mathbf{u}}$.

We consider the convex hulls of the connected components of $G_{\mathbf{u}}$.

Definition

(CTV2) A Markov polytope is the convex hull of the elements in a connected component of this graph.

Universal Markov and universal Gröbner basis

Theorem

(StWeZi 95) $\mathbf{u} \in L$ *is in the universal Gröbner basis of* A *if* $\mathbf{u}$ *is in the Graver basis of* A *and* $[\mathbf{u}^+, \mathbf{u}^-]$ *is an edge of the convex hull of all points in* $\mathcal{F}_u$.

We get the following characterization:

Theorem

(CTV2) An element $\mathbf{u}$ *of the universal Markov basis of* A *belongs to the universal Gröbner basis of* A *if and only if* $\mathbf{u}^+$ *and* $\mathbf{u}^-$ *are vertices of two different (Markov) polytopes.*

Example of Markov polytope

Example

Let A be the matrix of the previous example. Recall that $x_1^2x_2^2x_3x_4 - x_5x_6x_7x_8$ is in the universal Markov basis of I_A but not in the universal Gröbner basis of I_A. Let $\mathbf{u} = (2, 2, 1, 1, -1, -1, -1, -1) \in L$. Then $|\mathcal{F}_{\mathbf{u}}| = 7$ and $\mathcal{F}_{\mathbf{u}} =$

$$\{(3, 3, 0, \ldots, 0), u^+, (1, 1, 2, 2, 0, 0, 0, 0), (0, 0, 3, 3, 0, 0, 0, 0)\}$$

$$\cup \{(0, \ldots, 0, 2, 2, 0, 0), u^-, (0, \ldots, 0, 2, 2)\}$$

The graph $G_{\mathbf{u}}$ has two connected components.

The Markov polytopes are line segments: $\mathbf{u}^+$ and $\mathbf{u}^-$ are not vertices of their Markov polytopes.

Universal Markov basis for positive toric ideals

Consider the toric ideal I_A such that $L(A) \cap \mathbb{N}^n = \{\mathbf{0}\}$. We have the following inclusions:

$$\mathcal{S}(I_A) \subseteq \mathcal{M}(I_A) \subseteq \mathcal{G}(I_A).$$

- (St95): $\mathcal{G}(I_A)$ is the subset of $L(A)$ whose elements have no proper conformal decomposition.
- (HS2005+CTV3): $\mathcal{S}(I_A)$ is the subset of $L(A)$ whose elements have no proper semiconformal decomposition.
- (CTV3): $\mathcal{M}(I_A)$ is the subset of $L(A)$ whose elements have no proper strongly semiconformal decomposition.

Markov complexity $m(A)$

Let A be an arbitrary integer matrix.

- Santos, Sturmfels 2003: $g(A)$ is equal to the maximum 1-norm of an element in $\mathcal{G}(\mathcal{G}(A))$. Thus $g(A)$ is finite.
- Santos, Sturmfels 2003: $m(A) \leq g(A)$.
- Hoşten, Sullivant 2005: $m(A) \geq$ the maximum 1-norm of any element in $\mathcal{G}(\mathcal{S}(A))$.

How to compute $m(A)$ in general?

So far a mystery!!!

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Markov complexity for monomial curves in $\mathbb{A}^3$

Theorem

(CTV3) Let $A = \{n_1, n_2, n_3\}$ be a set of positive integers with $\gcd(n_1, n_2, n_3) = 1$. Then $m(A) = 2$ if A is complete intersection, and $m(A) = 3$ if A is not complete intersection. Moreover, for any $r \geq 2$ we have $\mathcal{M}(A^{(r)}) = \mathcal{S}(A^{(r)})$

Theorem

(CTV3) Let $A = \{n_1, n_2, n_3\}$ such that $\gcd(n_1, n_2, n_3) = 1$ and $d_{ij} = \gcd(n_i, n_j)$ for all $i \neq j$. Then

$$g(A) \geq \frac{n_1}{d_{12}d_{13}} + \frac{n_2}{d_{12}d_{23}} + \frac{n_3}{d_{13}d_{23}}.$$

In particular, if n_1, n_2, n_3 are pairwise prime then $g(A) \geq n_1 + n_2 + n_3$.

Examples

Examples

(a) Let $A = \{3, 4, 5\}$. Computations with 4ti2 show that the maximum 1-norm of the elements of $\mathcal{G}(\mathcal{G}(A))$ is $12 = 3 + 4 + 5$ and thus $g(A)$ equals the the lower bound of the theorem.

(b) Let $A = \{2, 3, 17\}$. Computations with 4ti2 show that the maximum 1-norm of the elements of $\mathcal{G}(\mathcal{G}(A))$ is 30 and thus $g(A) = 30$, while the lower bound of the theorem is $22 = 2 + 3 + 17$.

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(b) Let $A = \{2, 3, 17\}$. Computations with 4ti2 show that the maximum 1-norm of the elements of $\mathcal{G}(\mathcal{G}(A))$ is 30 and thus $g(A) = 30$, while the lower bound of the theorem is $22 = 2 + 3 + 17$.